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The problem of wave propagation in a random medium is formulated in terms of Feynman's path integral. It turns out to be a powerful calculational tool. The emphasis is on propagation conditions where the rms (multiple) scattering angle is small but the log-intensity fluctuations are of order unity - the so-called saturated regime. It is shown that the intensity distribution is then approximately Rayleigh with calculable corrections.

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of the wave field is explicitly shown to be valid whenever the rms multiple scattering angle is small. It is then shown that in the saturated regime the third and higher moments can be obtained from the first two by the rules of Gaussian statistics. There are small calculable corrections to the Gaussian law leading to "coherence tails". Correlations between waves of different frequencies and the physics of pulse propagation are studied in detail. Finally it is shown that the phenomenon of saturation is physically due to the appearance of many Fermat paths satisfying a perturbed ray equation.

For clarity of presentation much of the paper deals with an idealized medium which is statistically homogeneous and isotropic and is characterized by fluctuations of a single typical scale size. However, the extension to inhomogeneous, anisotropic and multiple scale media is given. The main results are summarized at the beginning of the paper.

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In an isotropic medium, the local or Markov approximation which is commonly used to compute first and second (at arbitrary space-time separation) moments of the wave field is explicitly shown to be valid whenever the rms multiple scattering angle is small. It is then shown that in the saturated regime the third and higher moments can be obtained from the first two by the rules of Gaussian statistics. There are small calculable corrections to the Gaussian law leading to "coherence tails". Correlations between waves of different frequencies and the physics of pulse propagation are studied in detail. Finally it is shown that the phenomenon of saturation is physically due to the appearance of many Fermat paths satisfying a perturbed ray equation.

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1. Introduction and Summary of Results

The problem of propagation of waves in a random medium appears in a number of areas of research and applied science. Some examples are atmospheric optics, radio astronomy and underwater sound. The problem is furthermore an old one which has been studied extensively. The earlier work (summarized in the monographs of Tatarskii[1] and Chernov[2]) employed the Rytov approximation. In this approximation the logarithm of the amplitude is computed using first order perturbation theory. The Rytov method is applicable whenever the intensity fluctuations are small. When the wavelength is small it reduces to first order geometric optics or WKB. More recently, a different approximation which reduces the problem to a Markov process has lead to considerable progress in cases where the intensity fluctuations are not small. This method is explained in Tatarskii's second book[3] and in two excellent reviews of the recent literature.[4,5] Nevertheless, important problems remain. In particular, there does not exist a global view of what is going on in the so-called saturated regime where the intensity fluctuations are important.

In this paper Feynman's path integral[6] is applied to the problem of wave propagation in a random medium. It provides a natural and systematic method for attacking the problem, especially when the intensity fluctuations are large and the Rytov approximation fails. The path integral is widely used in quantum mechanics and statistical mechanics but it is expected that many readers will not be familiar with it, thus the paper is meant to be self-contained. The reader who desires further background information on path integrals will do well to consult the book of Feynman and Hibbs.[6]

Because some readers will not be familiar with path integrals there are some pecularities in the organization of this paper. In real situations, random media are often statistically inhomogeneous or anisotropic and

frequently have a power law spectrum in the scale size of fluctuations. Path integrals are capable of handling all these complications. (In fact the author first developed the method for propagation of sound in the ocean[7], a problem which has these complications and more.) However, it is vastly easier to explain the path integral method for an idealized medium which is statistically homogeneous and isotropic and whose fluctuations are characterized by a single[8] typical scale size L (small compared to the distance R of propagation). The bulk of the paper is therefore devoted to a study of this idealized situation. Once this has been done the transition to realistic media is relatively simple. However, this manner of presentation has a defect for which only an apology can be offered. Because of the temporary restriction to a single scale size L, results which are directly applicable to atmospheric optics do not appear until late in the paper (specifically, Secs. (7) and (8)). Finally, to illustrate the power of the path integral method (and, hopefully, motivate the reader), a number of results for the idealized problem will be summarized below. The translation of these results to more complicated cases is generally straightforward: the details are given in the text. Listing the results will require the definition of some symbols. This will (temporarily) be done in terms of the idealized problem and the reader who has worked on propagation in a random medium will find that they are familiar objects; e.g., Tatarskii's phase structure function D. For other readers, the motivation for these definitions will become apparent in Secs. (2) and (3).

Actually, there are two distinct kinds of problems of propagation in a random medium, corresponding to whether the scattering angles, single and/or multiple, are large or small. If the fluctuations are weak so that a single scattering approximation (Born approximation) applies there is little distinction between the two cases. However in a multiple scattering regime,

which is the case of interest here, the two kinds of problems are very different. This is illustrated in Fig. (1). The considerations of this paper will be restricted to situations where the single and multiple scattering angles are small. This is sufficient to cover the applications mentioned above. The large angle multiple scattering situation is like a problem in radiative transport and is most efficiently treated by other methods.

It will be assumed that the problem can be reduced to a scalar wave equation with an index of refraction $n(\vec{x},t)$ which may depend on the frequency $\omega = ck$. In a homogeneous medium $\langle n \rangle$ is a constant and for waves of a fixed frequency can be set equal to unity. Defining

$$\mu(\vec{x},t) = 1 - n(\vec{x},t) \tag{1.1}$$

μ will be taken to have a zero mean and a covariance

$$\langle \mu(\vec{x},t)\ \mu(\vec{x}',t') \rangle = \rho(|\vec{x} - \vec{x}'|, t - t'). \tag{1.2}$$

It will be further assumed that either μ is a Gaussian[9] random field or that $kL\langle \mu^2 \rangle^{\frac{1}{2}}$ is small, in which case the distribution need not be specified.

Let the two dimensional vector $\vec{r}_o = (x_o, y_o)$ label the location of a point source[10] in the plane $z = 0$. Then in a plane of constant $z > 0$, the signal will be $E(z,\vec{r},\vec{r}_o,t)$ where $\vec{r} = (x,y)$ specifies the transverse coordinates of the observation point. The total range of propagation will be denoted by R and for $|\vec{r}|$, $|\vec{r}_o| \ll R$ and a CW source it is useful to define a complex envelope $\mathcal{E}$ by

$$E(z,\vec{r},\vec{r}_o,t) = \mathrm{Re}\left[\mathcal{E}(z,\vec{r},\vec{r}_o,t)\ e^{i(kz - \omega t)}\right] \tag{1.3}$$

The time dependence of $\mathcal{E}$ is due to fluctuations in the medium. It will be assumed that the full wave equation for $\mathcal{E}$ can be approximated by the parabolic wave equation[3-7,11]

$$\left(i\frac{\partial}{\partial z} + \frac{1}{2k}\nabla^2 - k\mu(\vec{r},z,t)\right)\mathcal{E}(\vec{r},\vec{r}_o,z,t) = 0 \qquad (1.4)$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ plus a boundary condition at $z = 0$

$$\mathcal{E} \rightarrow (4\pi z)^{-1} \exp\left[\frac{ik(\vec{r} - \vec{r}_o)^2}{2z}\right] \qquad (1.5)$$

If L and T are the characteristic lengths and times over which μ changes, the validity condition for Eq. (1.4) are (i) $kL \gg 1$, (ii) $kL \ll \omega T$ and (iii) that the rms multiple scattering angle $(\langle\mu^2\rangle R/L)^{\frac{1}{2}}$ should be small.

Feynman's path integral gives the solution to the parabolic wave equation in terms of a (strictly speaking) infinite dimensional integral. It turns out that this integral can be studied in almost exactly the same way as Mercier[12] originally attacked the phase screen integral. The result is that propagation in a statistically homogeneous medium is very similar to the phase screen problem. This will continue to be true in rather general inhomogeneous media, of which the phase screen is a special case.

In order to indicate what can be learned from the path integral it is necessary to review some known features of propagation in a random medium. The qualitative character of $\mathcal{E}$ is determined by two parameters Φ and Ω defined by

$$\Phi^2 = k^2\left\langle\left(\int_0^R \mu(\vec{e}_z\, z, t)\,dz\right)^2\right\rangle$$

$$= k^2 R\int_{-\infty}^{\infty}\rho(|z|,0)\,dz + O(L/R) \qquad (1.6)$$

where $\vec{e}_z$ is a unit vector in the z direction and in the second line it has been assumed that $R \gg L$ and

$$\Omega = 6kL^2/R. \tag{1.7}$$

The parameter Φ is just the rms phase fluctuation as computed in first order geometric optics[1] and serves as a measure of the strength of the fluctuations. The other parameter Ω is essentially the square of the ratio of the scale size L to the extent of a Fresnel zone. As shown in Figure 2, if Φ is less than one <u>or</u> less than Ω, then the Rytov[1-5] approximation is valid. In the region where the Rytov approximation is valid, the problem can be considered to have been solved years ago. The intensity fluctuations (scintillations) are small and the relation between $\mathcal{E}$ and μ is simple and direct. Also, as shown in Figure 2, when both Φ and Φ/Ω are greater than unity, the fluctuations in $\mathcal{E}$ saturate. In particular, the variance of $\ln |\mathcal{E}|^2 = \ln I$ approaches a constant of order unity and the properties of $\mathcal{E}$ are determined more by statistical considerations than by the detailed properties of μ.

Path integral methods have nothing new to add when the Rytov approximation is valid. The considerations of this paper will therefore be restricted to the saturated regions. There is then a small parameter $\alpha = \Omega/\Phi$ whose order of magnitude[14] is

$$\alpha \sim \frac{6L^{3/2}}{R^{3/2}\langle\mu^2\rangle^{\frac{1}{2}}} \tag{1.8}$$

The path integral allows the calculation of any moment of $\mathcal{E}$ as an asymtotic series in α. The result is that $\mathcal{E}$ is uniformly distributed in phase and that the moments of intensity $I \equiv |\mathcal{E}|^2$ are given by

$$\langle I^n \rangle = n!\langle I\rangle^n[1 + \tfrac{1}{2}n(n-1)C\alpha + O(\alpha^2)] \tag{1.9}$$

where C is a calculable constant of order unity whose precise value depends on the spectrum of μ. In the limit $\alpha = 0$ the distribution is therefore Rayleigh[15] with

$$P(I) = \frac{1}{\langle I\rangle} \exp\left[-\frac{I}{\langle I\rangle}\right] \tag{1.10}$$

However, the correction grows with n and cannot be neglected for $n \gtrsim (2/\alpha)^{\frac{1}{2}}$. It follows that there must be significant deviations from a Rayleigh distribution when $I/\langle I\rangle$ is greater than $\sim (2/\alpha)^{\frac{1}{2}}$.

In addition to the distribution of $\mathcal{E}$, one also wants to know the coherences in space and time. Recent work on coherences has been greatly facilitated by the observation[3-5] that under certain conditions the problem can be replaced by a simpler local or Markov one where, in effect, one makes the replacement

$$\rho(\vec{x},t) \rightarrow \delta(z)\hat{\rho}(|\vec{r}|,t) \tag{1.11}$$

with $\vec{r} = (x,y)$ and

$$\hat{\rho}(|\vec{r}|,t) = \int_{-\infty}^{\infty} \rho\left((\vec{r}^{\,2} + z^2)^{\frac{1}{2}}, t\right) dz \tag{1.12}$$

Note that Φ^2 is equal to $k^2 R\hat{\rho}(0,0)$ and it will be convenient to use the function $\hat{\rho}$ to define T and L by the expansion

$$k^2 R\hat{\rho}(|\vec{r}|,t) = \Phi^2\left(1 - \frac{\vec{r}^{\,2}}{2L^2} - \frac{t^2}{2T^2} + \ldots\right) \tag{1.13}$$

Within factors of order unity, the L and T so defined will be equal to the length and time over which the original covariance ρ is non-vanishing.

It has been pointed out by several authors[3-5] that in the Markov approximation the coherence of $\mathcal{E}^*$ and $\mathcal{E}$ can be computed exactly. It is

$$\frac{\langle \mathcal{E}^*(\vec{r}\,',\vec{r}_o\,',t')\mathcal{E}(\vec{r},\vec{r}_o,t)\rangle}{\mathcal{E}_o^*(\vec{r}\,',\vec{r}_o\,')\mathcal{E}_o(\vec{r},\vec{r}_o)} = \exp\left[-\tfrac{1}{2}D(\vec{r}-\vec{r}\,',\vec{r}_o-\vec{r}_o\,',t-t')\right] \tag{1.14}$$

where

$$\mathcal{E}_o(\vec{r},\vec{r}_o) = \langle I\rangle^{\frac{1}{2}} \exp\left[ik\frac{(\vec{r} - \vec{r}_o)^2}{2R}\right] \tag{1.15}$$

and D is the phase structure function of first order geometric optics[1]

$$D(\vec{r},\vec{r}_o, t) = 2k^2R\int_0^1[\hat{\rho}(0,0) - \hat{\rho}(|u\vec{r} + (1 - u)\vec{r}_o|, t)]du \tag{1.16}$$

The phase structure function always appears in an exponential and in the saturated region where Φ^2 is large, D can be approximated by an expansion in $\vec{r},\vec{r}_o$ and t

$$D(\vec{r},\vec{r}_o,t) = \Phi^2\left[\frac{\vec{r}^2 + \vec{r}_o^2 + \vec{r}\cdot\vec{r}_o}{3L^2} + \frac{t^2}{T^2} + \dots\right] \tag{1.17}$$

Coherences are then characterized by two parameters Φ/T and Φ/L.

The literature is somewhat confusing as to the validity conditions for Eq. (1.14). It turns out that the approximation leading to Eq. (1.14) has a very simple interpretation in the path integral formalism. In the next section is will become evident that for the isotropic medium under consideration Eq. (1.14) is valid as long as the parabolic wave equation is valid. From the path integral one can actually compute the first correction to Eq. (1.14). It is of order of the rms multiple scattering angle $(R/L)^{\frac{1}{2}}\langle\mu^2\rangle^{\frac{1}{2}}$ which must be small if the parabolic wave equation is valid.

For small α the path integral also allows the calculation of $\langle\mathcal{E}^*(\vec{r},\vec{r}_o,t)\mathcal{E}^*(\vec{r}',\vec{r}_o',t')\mathcal{E}(\vec{r}'',\vec{r}_o'',t'')\mathcal{E}(\vec{r}''',\vec{r}_o''',t''')\rangle$ and more generally an arbitrary 2n-th order moment. In the limit $\alpha = 0$, the real and imaginary parts of $\mathcal{E}$ are jointly Gaussian. To see the use of this result, let us consider a typical question of practical interest. Take a fixed source and receiver so that $\mathcal{E}$ is a function only of time and suppose that at t = 0

$\mathcal{E}/\mathcal{E}_0$ is known to have a value η. An interesting practical question is then what is the probability $P(\eta')$ that $\mathcal{E}(t)/\mathcal{E}_0$ will take on the value η'. Since $\mathcal{E}$ has a Gaussian distribution, $P(\eta')$ is simply

$$P(\eta') = \frac{\exp\left[-\dfrac{|\eta' - e^{-\frac{1}{2}D(t)}\eta|^2}{1 - e^{-D(t)}}\right]}{\pi\left(1 - e^{-D(t)}\right)} \tag{1.18}$$

where $D(t) = D(\vec{0},\vec{0},t) \approx \Phi^2(t/T)^2$. The qualitative behavior of $P(\eta')$ is indicated in Figure 3. It is evident that the signal stays in one quadrant of the complex plane and is therefore coherent over a time of order T/Φ. A further property of Gaussian statistics and a covariance of the form $\exp\,[-\frac{1}{2}(\Phi t/T)^2]$ is that the signal will move in a straight line for times less than $\sim T/\Phi$. One can ask the more general question of given that $\mathcal{E}(\vec{r},\vec{r}_0,t)/\mathcal{E}_0(\vec{r},\vec{r}_0)$ is equal to η, what is the probability that $\mathcal{E}(\vec{r}\,',\vec{r}_0\,',t')/\mathcal{E}_0(\vec{r}\,',\vec{r}_0\,')$ will be equal to η'. The result is just Eq. (1.18) with $D(t)$ replaced by $D(\vec{r} - \vec{r}\,', \vec{r}_0 - \vec{r}_0\,', t - t')$.

As stated above the Gaussian statistics leading to $P(\eta')$ are obtained by computing moments. Again the approximation scheme breaks down for moments of order $(2/\alpha)^{1/2}$ and Eq. (1.18) is valid only for $|\eta|$ and $|\eta'|$ less than $(2/\alpha)^{1/4}$. Actually the order α corrections to any moment are calculable. They are most important for intensity correlations where they lead to coherence tails of order α which are small but fall much less rapidly than e^{-D}.

The path integral also provides a simple method for calculating the correlation between waves of different frequencies. In the saturated region where $\Phi > 1$ the result is, for $|\omega-\omega'|$ small compared to $\bar{\omega} \equiv \frac{1}{2}(\omega+\omega')$

$$\frac{\langle \mathcal{E}^*(\omega')\mathcal{E}(\omega)\rangle}{\mathcal{E}_0^*(\omega')\,\mathcal{E}_0(\omega)} = \exp\left[-\frac{1}{2}\left(\frac{\omega-\omega'}{\omega_g}\right)^2\right]\Lambda(\omega-\omega') \tag{1.19}$$

where ω_g is

$$\omega_g^{-2} = \left\langle \left(\int_0^R \frac{d}{d\omega} \left(k\mu(\vec{e}_z z,t) \right) dz \right)^2 \right\rangle \tag{1.20}$$

and for a single scale medium

$$\Lambda(\omega) = \frac{\left(\frac{6i\omega}{\omega_o \alpha} \right)^{1/2}}{\sin\left(\frac{6i\omega}{\omega_o \alpha} \right)^{1/2}} \tag{1.21}$$

with $\omega_o^{-2} = c_g^{-2} R\, \hat{\rho}(0,0)$ where c_g is the unperturbed group velocity. For a non-dispersive medium $\omega_o = \omega_g$. When α is very small the second factor on the right-hand side of Eq. (1.19) falls much more rapidly than the first one. The first factor $\exp\left[-\frac{1}{2} \left(\frac{\omega-\omega'}{\omega_g} \right)^2 \right]$ can then be replaced by unity. In the limit $\alpha = 0$ the higher order correlations in frequency are Gaussian. One can then obtain probability distributions in frequency from Eq. (1.18) with $\exp[-D/2]$ replaced by the right hand side of Eq. (1.19) and e^{-D} replaced by its absolute value squared. It is worth noting that a first order geometric optics calculation misses the second and dominant factor on the right hand side of Eq. (1.19) and therefore vastly overestimates the range of coherence in frequency.

It can be seen from the path integral that saturation corresponds to the appearance of multiple Fermat paths which satisfy a perturbed ray equation. The signal tends to propagate along these Fermat paths and because there are many of them, they interfere and produce Gaussian statistics. They will become manifest in an experiment with a pulsed source where the received signal will tend to show several arrivals. These multiple Fermat paths are responsible for the factor Λ in Eq. (1.19).

With one exception these results can easily be extended to statistically inhomogeneous or anisotropic media and to media with multiple scales. The exception is that $\Lambda(\omega)$ defined in Eq. (1.19) cannot be computed for certain multiple scale media. Actually, the path integral yields further information in the case of multiple scale media. It appears to be only partially understood[3-5] that in this case there are two distinct saturated regimes. An examination of the path integral shows that there are indeed two, one of which (the fully saturated regime) is analogous to the saturated regime in single scale media and another one (the partially saturated regime) is new. Many experiments in atmospheric optics lie in the partially saturated regime and this case is treated in some detail (Sec. 8). The fundamental distinction between the fully and partially saturated regimes shows up in correlations between waves of different frequency. In the fully saturated regime the real and imaginary parts of $\mathcal{E}(\omega)$ are jointly Gaussian random variables. For partial saturation $\mathcal{E}(\omega)$ acts like a random phase times a Gaussian object. A consequence is that propagation of narrow pulses is qualitatively different in the two regimes. Depending on the medium there may be further qualitative differences between full and partial saturation.

The detailed organization of the paper is as follows. Secs. (2)-(6) and Apps. (A) and (B) are devoted to the idealized homogeneous, isotropic medium with a single scale size. In Sec. (2) the path integral is introduced and applied to the calculation of the first and second moments. App. (A) contains the calculation of the error in Eq. (1.14). Sec. (3) is devoted to the calculation of higher moments when α is small and Sec. (4) summarizes the statistics of $\mathcal{E}$ in the limit $\alpha = 0$. Special attention is given to statistics in frequency and pulse propagation. The corrections to the limiting statistics are derived in App. (B) and discussed in Sec. (5). The appearance

of multiple Fermat paths is demonstrated in Sec. (6). Media with multiple scales are introduced in Sec. (7) and the distinction between full and partial saturation is made. In the fully saturated case there is a simple modification of the results for a single scale medium (Table 2). The partially saturated regime is more difficult. Sec. (8) is devoted to partial saturation in a medium like that encountered in atmospheric optics. App. (C) contains some calculations relevant to Sec. (8) and App. (D) discusses some other kinds of multiple scale media. Methods for handling inhomogeneous and anisotropic media are given in Sec. (9) and Apps. (E) and (F).

2. FIRST AND SECOND MOMENTS FROM THE PATH INTEGRAL

Feynman[6] pointed out that the solution to Eq. (1.4) with the boundary condition in Eq. (1.5) is given by an infinite dimensional integral. It is defined as the limit of a finite dimensional integral with 2n - 2 integration variables corresponding to the Cartesian components of n - 1 two-dimensional vectors $\vec{r}_j$, j = 1, 2, ... n - 1. With the convention that $\vec{r}_j|_{j=0}$ and $\vec{r}_j|_{j=n}$ are the source $\vec{r}_o$ and receiver $\vec{r}$ coordinates, Feynman's integral is

$$\mathcal{E}(\vec{r},\vec{r}_o,t) = \lim_{n\to\infty} \frac{i}{2k}\int\left(\prod_{j=1}^{n-1} d^2r_j\right)\left(\frac{kn}{2\pi iR}\right)^n \exp\left[\frac{ikR}{n}\sum_{j=1}^{n}\left(\frac{n^2}{2}\left(\frac{\vec{r}_j-\vec{r}_{j-1}}{R}\right)^2 - \mu(\vec{r}_j+\vec{e}_z z_j,t)\right)\right] \tag{2.1}$$

where each component of $\vec{r}_j$, j = 1, ... n - 1, is integrated over the range $-\infty$ to $+\infty$ and $z_j = jR/n$. In μ, $\vec{r}_j$ is understood to be a vector in the (x,y) plane and $\vec{e}_z$ is a unit vector in the z direction. At each point $\vec{r}_1, \vec{r}_2 \ldots \vec{r}_{n-1}$ in the integration volume, the n - 1 points in space $(\vec{r}_1,z_1)$, $(\vec{r}_2,z_2)$, ... $(\vec{r}_{n-1},z_{n-1})$ can be thought of as discrete points along a path $\vec{r}(z)$ connecting $(\vec{r}_o,0)$ to $(\vec{r},R)$ with $\vec{r}_j = \vec{r}(z_j)$, see Fig. (4). In this sense Feynman's

integral is an integral over paths. Associating R/n with a differential increment dz in range the argument of the exponential has a continuum limit

$$\frac{ikR}{n} \sum_{j=1}^{n} \left(\frac{n^2}{2} \left(\frac{\vec{r}_j - \vec{r}_{j-1}}{R} \right)^2 - \mu(\vec{r}_j + \vec{e}_z z_j, t) \right) \rightarrow ik \int_0^R \left[\frac{1}{2} \left(\vec{r}\,'(z) \right)^2 - \mu\left(\vec{r}(z) + \vec{e}_z z, t \right) \right] dz \tag{2.2}$$

where $\vec{r}\,' \equiv d\vec{r}/dz$. The path integral for $\mathcal{E}$ can then be schematically written as

$$\mathcal{E}(\vec{r},\vec{r}_o,t) = \frac{i}{2k} \int d(\text{paths}) \exp\left[ik \int_0^R \left[\frac{1}{2}\left(\vec{r}\,'(z)\right)^2 - \mu\left(\vec{r}(z) + \vec{e}_z z, t\right) \right] dz \right] \tag{2.3}$$

where the integration is over all paths connecting $(\vec{r}_o, 0)$ to $(\vec{r}, R)$ and the volume element in path space d(paths) is the coefficient of the exponential in Eq. (2.1).

We will be computing averages of products of path integrals and the following formula will be needed. Let $\vec{r}_n(z)$, $n = 1,2,\ldots$ be some set of paths and $\xi_n = \pm 1$ corresponding phases. Then if <u>either</u> μ is a Gaussian random field <u>or</u> $kL \langle \mu^2 \rangle^{1/2} << 1$ and its statistics are arbitrary, it is well known[1-6] that

$$\left\langle \exp\left[-ik \sum_n \xi_n \int_0^R \mu\left(\vec{r}_n(z) + \vec{e}_z z, t_n \right) dz \right] \right\rangle$$

$$= \exp\left[-\frac{k^2}{2} \sum_{m,n} \xi_m \xi_n \int_0^R \int_0^R \rho\left(\sqrt{\left(\vec{r}_n(z) - \vec{r}_m(z') \right)^2 + (z'-z)^2}\, , t_n - t_m \right) dz\,dz' \right] \tag{2.4}$$

As a first application of the path integral we can compute $\langle \mathcal{E} \rangle$. This will not turn out to be a particularly interesting quantity but the calculation is simple and it will show how path integrals work and where the Markov approximation comes in. Bringing the average inside the path integral and using Eq. (2.4) yields

$$\langle \mathcal{E} \rangle = \frac{i}{2k} \int d(\text{paths}) \exp\left[\frac{ik}{2}\int_0^R (\vec{r}\,')^2 dz - \frac{k^2}{2}\int_0^R\int_0^R \rho\left(\sqrt{\left(\vec{r}(z) - \vec{r}(z')\right)^2 + (z - z')^2}, 0\right) dz dz'\right] \tag{2.5}$$

The Markov approximation now appears as follows. The parabolic wave equation assumes that the normals to the wave fronts point in directions that are close to the z-axis. In terms of the path integral this means that for the important paths $|\vec{r}\,'| = |d\vec{r}/dz|$ must be small. It then follows that for important paths $\left(\vec{r}(z) - \vec{r}(z')\right)^2 + (z - z')^2 \approx |z - z'|^2$ and Eq. (2.5) becomes

$$\langle \mathcal{E} \rangle = \frac{i}{2k} \exp\left[-\frac{k^2}{2}\int_0^R\int_0^R \rho(|z - z'|, 0) dz dz'\right] \int d(\text{paths}) \exp\left[\frac{ik}{2}\int_0^R (\vec{r}\,')^2 dz\right] \tag{2.6}$$

The remaining path integral is just the path integral for $\mathcal{E}_o$ and for $R \gg L$ the double integral over ρ can be replaced by $R\hat{\rho}(0,0)$. The final result is then

$$\langle \mathcal{E} \rangle = \mathcal{E}_o \exp\left[-\tfrac{1}{2}\Phi^2\right] \tag{2.7}$$

This is the usual formula obtained in the Markov approximation.[3-5] What we have seen here is that this approximation has a very natural interpretation in terms of the path integral and that it is valid as long as the parabolic wave equation is valid.

Since Φ^2 is large in the saturated region $\langle \mathcal{E} \rangle$ is exponentially small and therefore not particularly interesting. The same is true for $\langle \mathcal{E}\mathcal{E} \rangle$ and its complex conjugate $\langle \mathcal{E}^*\mathcal{E}^* \rangle$. The path integral for $\langle \mathcal{E}\mathcal{E} \rangle$ will be a double path integral over two paths $\vec{r}_1(z)$ and $\vec{r}_2(z)$ and will contain a factor

$$\exp\left[-\frac{k^2}{2}\int_0^R\int_0^R\left[\rho\left(\sqrt{\left(\vec{r}_1(z)-\vec{r}_1(z')\right)^2+(z-z')^2},0\right)+\right.\right.$$

$$\left.\left.\rho\left(\sqrt{\left(\vec{r}_2(z)-\vec{r}_2(z')\right)^2+(z-z')^2},0\right)+2\rho\left(\sqrt{\left(\vec{r}_1(z)-\vec{r}_2(z')\right)^2+(z'-z)^2},t\right)\right]dzdz'\right]$$

(2.8)

where t is the time difference between the two $\mathcal{E}$'s in the average. This factor is of order exp $[-\Phi^2]$ in all important regions of path space and $\langle \mathcal{E}\mathcal{E} \rangle$ is exponentially small.

A more interesting quantity is $\langle \mathcal{E}^*(2)\mathcal{E}(1) \rangle$ where $\mathcal{E}(1)$ is a shorthand notation for $\mathcal{E}(\vec{r}_1,\vec{r}_{o1},t_1)$ and $\mathcal{E}^*(2)$ for $\mathcal{E}^*(\vec{r}_2,\vec{r}_{o2},t_2)$. The formula for this object is

$$\frac{1}{4k^2}\int d^2(\text{paths})\exp\left[\frac{ik}{2}\int_0^R\left[\left(\vec{r}_1{}'(z)\right)^2-\left(\vec{r}_2{}'(z)\right)^2\right]dz-V\right] \qquad (2.9)$$

where the path integral is a "double path integral" over two paths $\vec{r}_1(z)$ and $\vec{r}_2(z)$ connecting $(\vec{r}_{o1},0)$ to $(\vec{r}_1,R)$ and $(\vec{r}_{o2},0)$ to $(\vec{r}_2,R)$ respectively and

$$V=\frac{k^2}{2}\int_0^R\int_0^R\left[\rho\left(\sqrt{\left(\vec{r}_1(z)-\vec{r}_1(z')\right)^2+(z-z')^2},0\right)\right.$$
$$+\rho\left(\sqrt{\left(\vec{r}_2(z)-\vec{r}_2(z')\right)^2+(z-z')^2},0\right)$$
$$\left.-2\rho\left(\sqrt{\left(\vec{r}_1(z)-\vec{r}_2(z')\right)^2+(z-z')^2},t_1-t_2\right)\right]dzdz' \qquad (2.10)$$

There is now a region in path space where the integrand is not exponentially small. It is $\vec{r}_1(z) \approx \vec{r}_2(z)$ and almost all of the path integral will come from this region. As before, $\left(\vec{r}_1(z) - \vec{r}_1(z')\right)^2$ and $\left(\vec{r}_2(z) - \vec{r}_2(z')\right)^2$ can be neglected relative to $(z - z')^2$ and in the same spirit $\sqrt{\left(\vec{r}_1(z) - \vec{r}_2(z')\right)^2 + (z' - z)^2}$ can be approximated by $\sqrt{\left(\vec{r}_1(\bar{z}) - \vec{r}_2(\bar{z})\right)^2 + (z' - z)^2}$ where $\bar{z} = \frac{1}{2}(z + z')$. Then for $R \gg L$ the integral over $z - z'$ can be done and

$$V = \int_0^R d\left(\left|\vec{r}_1(z) - \vec{r}_2(z)\right|, t_1 - t_2\right) dz \qquad (2.11)$$

where

$$d\left(|\vec{r}|, t\right) = k^2 \left[\hat{\rho}(0,0) - \hat{\rho}\left(|\vec{r}|, t\right)\right] \qquad (2.12)$$

At this point it is convenient to change variables to paths $\vec{u}(z)$ and $\vec{v}(z)$

$$\vec{u}(z) = \tfrac{1}{2}\left(\vec{r}_1(z) + \vec{r}_2(z)\right) - \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2)\frac{z}{R} - \tfrac{1}{2}(\vec{r}_{o1} + \vec{r}_{o2})\left(1 - \frac{z}{R}\right)$$

$$\vec{v}(z) = \vec{r}_1(z) - \vec{r}_2(z) \qquad (2.13)$$

which satisfy the endpoint conditions $\vec{u}(0) = \vec{u}(R) = \vec{0}$ and $\vec{v}(0) = (\vec{r}_{o1} - \vec{r}_{o2})$, $\vec{v}(R) = (\vec{r}_1 - \vec{r}_2)$. From the finite form of the path integral in Eq. (2.1) it is clear that this change of variables is allowed and that the associated Jacobian is equal to unity. After integrating the first term in the exponential by parts and using the endpoint conditions, the path integral for $\langle \mathcal{E}^*(2)\mathcal{E}(1) \rangle$ becomes

$$\langle \mathcal{E}^*(2)\mathcal{E}(1) \rangle = \left(\frac{2\pi R}{k}\right)^2 \mathcal{E}_o^*(2)\, \mathcal{E}_o(1) \int d^2(\text{paths}) \exp\left[- ik \int_0^R \vec{u}(z) \cdot \vec{v}''(z) dz - \int_0^R d(|\vec{v}(z)|, t) dz\right] \qquad (2.14)$$

where $t = t_1 - t_2$. In analogy with the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy} dy = \delta(x) \tag{2.15}$$

the integral over the path $\vec{u}$ in Eq. (2.14) will produce a "δ-functional" which forces $\vec{v}''$ to vanish identically.[16] With the endpoint conditions given above $\vec{v}(z)$ must then be

$$\vec{v}(z) = (\vec{r}_1 - \vec{r}_2)\frac{z}{R} + (\vec{r}_{o1} - \vec{r}_{o2})\left(1 - \frac{z}{R}\right) \tag{2.16}$$

In d the path $\vec{v}$ can then be replaced by the right-hand side of Eq. (2.16) and the factor containing d then becomes just $\exp[-\frac{1}{2}D]$. The remaining path integral is equal to $(2\pi R/k)^{-2}$and the result[17] reproduces Eq. (1.14)

$$\frac{\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle}{\mathcal{E}_o^*(2)\mathcal{E}_o(1)} = \exp\left[-\tfrac{1}{2}D(1,2)\right] \tag{1.14$'$}$$

where D(1,2) is a shorthand notation for $D(\vec{r}_1 - \vec{r}_2, \vec{r}_{o1} - \vec{r}_{o2}, t_1 - t_2)$.

Appendix A contains an explicit calculation of the first correction to the Markov approximation for $\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle$. It is shown to be proportional to the rms multiple scattering angle $(\langle \mu^2\rangle R/L)^{\frac{1}{2}}$ which must be small if the parabolic wave equation is valid. Henceforth, all calculations will be done in this Markov approximation. The general prescription is that whenever $\rho\left(\sqrt{(\vec{r}_i(z) - \vec{r}_j(z'))^2 + (z - z')^2}, t_i - t_j\right)$ appears, it is to be replaced by $\delta(z - z')\hat{\rho}\left(\left|\vec{r}_i\left(\frac{z + z'}{2}\right) - \vec{r}_j\left(\frac{z + z'}{2}\right)\right|, t_i - t_j\right)$.

Turning now to the calculation of $\langle \mathcal{E}^*(\omega')\mathcal{E}(\omega)\rangle$, the path integral for this quantity will contain (with $k = k(\omega)$ and $k' = k(\omega')$)

$$\left\langle \exp\left[-ik\int_0^R \mu_\omega\left(\vec{r}_1(z) + \vec{e}_z z\right)dz + ik'\int_0^R \mu_{\omega'}\left(\vec{r}_2(z) + \vec{e}_z z\right)dz\right]\right\rangle \tag{2.17}$$

where the time dependence of μ has been suppressed and the subscript indicates that for a dispersive medium μ can depend on ω. Let us first

compute this average in the absence of dispersion. When μ_ω is independent of ω it is, in the Markov approximation

$$\exp\left[-\tfrac{1}{2}(k - k')^2 R\hat{\rho}(0,0) - kk'\int_0^R [\hat{\rho}(0,0) - \hat{\rho}(|\vec{r}_1(z) - \vec{r}_2(z)|,0)]dz\right] \tag{2.18}$$

For paths which make a significant contribution to the path integral, the second term in the argument of the exponential must be of order unity or less. In this term one can therefore approximate kk' by $\bar{k}^2$ where $\bar{k} = \frac{1}{2}(k + k')$. Generalizing to dispersive media, one finds that in the same approximation the result is just Eq. (2.18) with $R\hat{\rho}(0,0)$ replaced by $(c_g/\omega_g)^2$ where ω_g was defined in Eq. (1.20). The path integral will also contain a factor

$$\exp\left[\frac{ik}{2}\int_0^R \left(\dot{\vec{r}}_1(z)\right)^2 dz - \frac{ik'}{2}\int_0^R \left(\dot{\vec{r}}_2(z)\right)^2 dz\right] \tag{2.19}$$

which can be simplified by making an orthagonal transformation to paths $\vec{u}$ and $\vec{v}$ defined by

$$\vec{r}_1(z) = \vec{u}(z) - \frac{k'\vec{v}(z)}{k-k'}$$

$$\vec{r}_2(z) = \vec{u}(z) - \frac{k\vec{v}(z)}{k-k'} \tag{2.20}$$

After making this transformation the path integral factors into a product of integrals over $\vec{u}$ and $\vec{v}$. Upon dividing by $\mathcal{E}_o^*(\omega')\mathcal{E}_o(\omega)$ the integral over $\vec{u}$ cancels and the final result is, for $|\omega - \omega'|$ small compared to $\bar{\omega} = \frac{1}{2}(\omega+\omega')$,

$$\frac{\langle \mathcal{E}^*(\omega')\mathcal{E}(\omega)\rangle}{\mathcal{E}_o^*(\omega')\mathcal{E}_o(\omega)} = \exp\left[-\frac{1}{2}\left(\frac{\omega-\omega'}{\omega_g}\right)^2\right]\Lambda(\omega - \omega') \tag{2.21}$$

where

$$\Lambda(\omega-\omega') = \frac{\int d(\text{paths}) \exp\left[-\frac{i\bar{k}^2}{2(k-k')}\int_0^R \left(\vec{v}'(z)\right)^2 dz - \bar{k}^2\int_0^R [\hat{\rho}(0,0)-\hat{\rho}(\vec{v}(z),o)]dz\right]}{\int d(\text{paths}) \exp\left[-\frac{i\bar{k}^2}{2(k-k')}\int_0^R (\vec{v}'(z))^2 dz\right]}$$

(2.22)

In the saturated region where Φ is large $|\vec{v}(z)|$ will be very small for the important paths and the expansion

$$\hat{\rho}(0,0) - \hat{\rho}(|\vec{v}(x)|,0) \approx \tfrac{1}{2}\hat{\rho}(0,0)\left(\frac{\vec{v}(x)}{L}\right)^2 \tag{2.23}$$

can be used. The path integral for Λ is then

$$\Lambda(\omega - \omega') = \frac{\int d(\text{paths})\exp\left[-\frac{i\bar{k}^2}{2(k-k')}\int_0^R (\vec{v}'(z))^2 dz - \frac{\bar{k}^2\hat{\rho}(0,0)}{2L^2}\int_0^R (\vec{v}(z))^2 dz\right]}{\int d(\text{paths})\exp\left[-\frac{i\bar{k}^2}{2(k-k')}\int_0^R (\vec{v}'(z))^2 dz\right]}$$

(2.24)

This type of path integral was evaluated by Feynman and setting $k - k' = (\omega-\omega')/c_g$ it is equal to

$$\Lambda(\omega - \omega') = \frac{\left(6i\,\frac{\omega - \omega'}{\omega_o\alpha}\right)^{\frac{1}{2}}}{\sin\left(6i\,\frac{\omega - \omega'}{\omega_o\,\alpha}\right)^{\frac{1}{2}}} \tag{2.25}$$

where $\omega_o^{-2} = R\hat{\rho}(0,0)/c_g^2$ and $\alpha = 6\left(L^4/R^3\,\hat{\rho}(0,0)\right)^{\frac{1}{2}} = \Omega/\Phi$. Combining equations (2.21) and (2.25) yields Eq. (1.19). Some features of these correlations in ω were mentioned in the Introduction. We will return to their interpretation in Section VI.

Except for the explicit verification of the validity of the Markov approximation, the above results could be obtained by more familiar techniques which do not employ the path integral. The power of the path integral will become apparent in the next section when higher order moments are computed. They are extremely difficult to treat by the usual techniques.

3. HIGHER MOMENTS FOR SMALL α

When Φ is large, the average of any path integral will be exponentially small unless there is a region of path space where each path associated with an $\mathcal{E}$ is close to a path associated with an $\mathcal{E}^*$. Such a region does not exist for $\langle\mathcal{E}\rangle$ or $\langle\mathcal{E}^2\rangle$ and we have already seen that they are exponentially small. More generally, any moment with an unequal number of $\mathcal{E}$'s and $\mathcal{E}^*$'s will be vanishingly small.

Beyond $\langle\mathcal{E}^*(2)\mathcal{E}(1)\rangle$ the first nontrivial object is $\langle\mathcal{E}^*(4)\mathcal{E}(3)\mathcal{E}^*(2)\mathcal{E}(1)\rangle$. It is given by the quadruple path integral over four paths $\vec{r}_1(z)\ldots\vec{r}_4(z)$

$$\langle\mathcal{E}^*(4)\mathcal{E}(3)\mathcal{E}^*(2)\mathcal{E}(1)\rangle = (2k)^{-4}\int d^4(\text{paths})\exp\left[-\frac{ik}{2}\sum_{j=1}^{4}(-1)^j\int_0^R(\vec{r}_j{}'(z))^2 dz - M\right] \tag{3.1}$$

where

$$M = -\tfrac{1}{2}\sum_{i,j=1}^{4}(-1)^{i+j}\int_0^R d(|\vec{r}_i(z) - \vec{r}_j(z)|, t_i - t_j)dz \tag{3.2}$$

There are two regions of path space where M is of order unity or smaller. They are: (a) $|\vec{r}_1(z) - \vec{r}_2(z)| < L/\Phi$, $|\vec{r}_3(z) - \vec{r}_4(z)| < L/\Phi$ with the distance between pairs of paths arbitrary and (b) $|\vec{r}_1(z) - \vec{r}_4(z)| < L/\Phi$, $|\vec{r}_3(z) - \vec{r}_2(z)| < L/\Phi$ again with the distance between pairs of paths arbitrary. In region (a) where $|\vec{r}_1(z) - \vec{r}_2(z)|$ is of order L/Φ, the oscillating factor

$$\exp\left[\frac{ik}{2}\int_0^R\left[\left(\vec{r}_1'(z)\right)^2 - \left(\vec{r}_2'(z)\right)^2\right]dz\right] \sim \exp\left[-\frac{ik}{2}\int_0^R\left(\vec{r}_1(z) - \vec{r}_2(z)\right) \cdot \left(\vec{r}_1''(z) + \vec{r}_2''(z)\right)dz\right]$$

(3.3)

in the path integral will restrict $|\vec{r}_1''(z) + \vec{r}_2''(z)|$ to be of order $2\Phi/(kLR)$. For a typical path $|\vec{r}_1(z) + \vec{r}_2(z)|$ will then be roughly $\frac{1}{3}\left(\frac{R}{2}\right)^2|\vec{r}_1''(z) + \vec{r}_2''(z)| \sim \frac{\Phi R}{6kL}$. The centroid of the other pair $\vec{r}_3(z) + \vec{r}_4(z)$ will be restrained in a similar way. It follows that most paths will be such that the ratio of the distance between the pairs to the scale length L is roughly $\Phi R/(6kL^2) = \alpha^{-1}$, where α is the parameter defined in the Introduction. For small α the pairs are separated by many times L and therefore are uncorrelated. In region (a) M then reduces to

$$M \approx \int_0^R d(|\vec{r}_1(z) - \vec{r}_2(z)|, t_1 - t_2)dz + \int_0^R d(|\vec{r}_3(z) - \vec{r}_4(z)|, t_3 - t_4)dz$$

(3.4)

and in region (b) it becomes

$$M \approx \int_0^R d(|\vec{r}_3(z) - \vec{r}_2(z)|, t_3 - t_2)dz + \int_0^R d(|\vec{r}_1(z) - \vec{r}_4(z)|, t_1 - t_4)dz$$

(3.5)

Thus in each of the two important regions of path space, the quadruple path integral factors into the product of two double path integrals, each of which is precisely the integral encountered in the calculation of $\langle\mathcal{E}^*\mathcal{E}\rangle$. The result is that

$$\langle\mathcal{E}^*(4)\mathcal{E}(3)\mathcal{E}^*(2)\mathcal{E}(1)\rangle \approx \langle\mathcal{E}^*(4)\mathcal{E}(3)\rangle\langle\mathcal{E}^*(2)\mathcal{E}(1)\rangle + \langle\mathcal{E}^*(2)\mathcal{E}(3)\rangle\langle\mathcal{E}^*(4)\mathcal{E}(1)\rangle$$

(3.6)

where the two terms come from the two regions (a) and (b).

In Appendix B the error in Eq. (2.6) is obtained by computing the first correction. It is of order α and will be discussed in detail in Section V.

Generalizing to an arbitrary moment is easy. The general non-vanishing moment is $\left\langle \prod_{j=1}^{n} \mathcal{E}^*(j) \prod_{i=1}^{n} \mathcal{E}(i) \right\rangle$ and can be written as an integral over 2n paths $\vec{r}_j(z)$ and $\vec{r}_i(z)$. There will now be n! important regions of path space corresponding to the number of ways paths $\vec{r}_j(z)$ can be paired with the paths $\vec{r}_i(z)$. In each of these regions the 2n-tuple path integral can be approximated by a product of n double path integrals. Some simple combinatorics shows that the result will be as follows. Let $\bar{i}$ be a permutation of the indices i. For example, if n = 3 and the permutation is (1,2,3) → (3,1,2) then $\bar{1} = 3$, $\bar{2} = 1$ and $\bar{3} = 2$ or if the permutation is (1,2,3) → (2,1,3) then $\bar{1} = 2$, $\bar{2} = 1$ and $\bar{3} = 3$. With this notation

$$\left\langle \prod_{j=1}^{n} \mathcal{E}^*(j) \prod_{i=1}^{n} \mathcal{E}(i) \right\rangle \approx \sum_{\text{perms}} \prod_{i,j=1}^{n} \left\langle \mathcal{E}^*(j)\mathcal{E}(\bar{i}) \right\rangle \tag{3.7}$$

where the sum of over all n! possible permutations of the indices i.

The same result holds for correlations in frequency. Extending the notation $\mathcal{E}(j)$ to include a frequency label ω_j we have

$$\frac{\langle \mathcal{E}^*(j)\mathcal{E}(i)\rangle}{\mathcal{E}_o^*(j)\mathcal{E}_o(i)} = \exp\left[-\tfrac{1}{2}D(i,j) - \tfrac{1}{2}\left(\frac{\omega_i - \omega_j}{\omega_g}\right)^2\right] \Lambda(\omega_i - \omega_j) \tag{3.8}$$

which holds in the saturated region where only small values of $\omega_i - \omega_j$, $|\vec{r}_i - \vec{r}_j|$, etc. are interesting. The same construction that led to Eq. (3.7) for equal frequencies then shows that it holds for unequal frequencies as well.

The interpretation of Eq. (3.7) will be given in the next section. A final remark here is that the arguments leading to Eq. (3.7) do not depend on the validity of the Markov approximation. The latter is needed only when $\langle \mathcal{E}^*(1)\mathcal{E}(2)\rangle$ is explicitly evaluated.

4. THE STATISTICS OF $\mathcal{E}$ IN THE LIMIT $\alpha = 0$

The moments of Eq. (3.7) correspond to a complex Gaussian distribution. The probability that $\mathcal{E}(j)/\mathcal{E}_o(j)$ will be equal to η_j for $j = 1,\ldots n$ is then

$$P_n(\eta_1,\ldots\eta_n) = \left(\det[\pi M]\right)^{-1} \exp\left[-\sum_{i,j=1}^{n} \eta_i^* (M)^{-1}_{ij}\, \eta_j\right] \tag{4.1}$$

where the n by n matrix M is

$$M_{ji} = \frac{\langle \mathcal{E}^*(i)\mathcal{E}(j)\rangle}{\mathcal{E}_o^*(i)\mathcal{E}_o(j)} = \exp\left[-\tfrac{1}{2}D(i,j) - \tfrac{1}{2}\left(\frac{\omega_j - \omega_i}{\omega_g}\right)^2\right] \Lambda(\omega_j - \omega_i) \tag{4.2}$$

Eq. (1.10) corresponds to the special case n = 1 and Eq. (1.18) is obtained by dividing $P_2(\eta,\eta')$ by $P_1(\eta)$. The measure is $d^2\eta = d(\mathrm{Im}\eta)d(\mathrm{Re}\eta)$.

In principle, Eqs. (4.1) and (4.2) determine all the statistical properties of $\mathcal{E}$. For example, it follows from Gaussian statistics that for $\mathcal{E}(j) = A(j)e^{i\phi(j)}$ the correlations of amplitude and rate of phase $\dot{\phi} = \frac{d\phi}{dt}$ are [18]

$$\frac{\langle A(1)A(2)\rangle}{\langle I\rangle} = E(|M_{12}|) - \tfrac{1}{2}(1 - |M_{12}|^2)K(|M_{12}|) \tag{4.3}$$

where E and K are the complete elliptic integrals of the first and second kinds and

$$\begin{aligned} \langle\dot{\varphi}(1)\dot{\varphi}(2)\rangle &= -\frac{\ddot{D}(1,2)}{4}\,\ell n(1 - |M_{12}|) \\ &= -\tfrac{1}{2}\frac{\Phi^2}{T^2}\,\ell n(1 - |M_{12}|) \end{aligned} \tag{4.4}$$

where in the second line the expansion of D in Eq. (1.17) has been used. Eq. (4.4) can be extended to the derivatives of φ with respect to $\vec{r}$ and $\vec{r}_o$ in the obvious way. Intensity correlations $I(j) = |A(j)|^2$ are

simpler with

$$\langle I(1)I(2)\rangle = \langle I\rangle^2 + |\langle \mathcal{E}^*(1)\mathcal{E}(2)\rangle|^2 \tag{4.5}$$

The appearance of Gaussian statistics in frequency is somewhat unfamiliar. To see what it implies, let us compare the saturated regime to a simplier unsaturated one. When simple first order geometric optics applies and the medium is nondispersive, $\mathcal{E}(\omega)$ is equal to $\mathcal{E}_o(\omega)e^{i\omega\hat{\tau}}$ where $\hat{\tau}$ is a fluctuating time shift independent of ω. Under such propagation conditions, the statistics in ω are essentially trivial. The envelope $\mathcal{E}(\omega)$ fluctuates but does so in such a way that at a fixed time when $\hat{\tau}$ has a definite value a knowledge of $\mathcal{E}$ at one value of ω determines $\mathcal{E}$ for all ω. Another way to say the same thing is that a pulse will be subjected to a random time shift but will not be distorted in shape. For propagation in the saturated regime the statistics of $\mathcal{E}(\omega)$ are non-trivial and things are completely different. At one fixed time a knowledge of $\mathcal{E}(\omega)$ at one ω yields only statistical information about $\mathcal{E}$ at nearby frequencies. Correspondingly, the medium will distort a pulse in a way that is predictable only statistically. A peculiarity is that $\langle \mathcal{E}^*(\omega')\mathcal{E}(\omega)\rangle$ and has a phase corresponding to an average retardation.

The above remarks about $\mathcal{E}(\omega)$ are most easily made quantitative in terms of pulse propagation. It is worth going into this in some detail both because the physics is interesting and because it will connect with the Fermat paths of Sec. (6). For simplicity the unperturbed medium will be assumed to be nondispersive with $\omega = ck$. Let the transmitted signal be $f_o(\tau) = \int e^{-i\omega\tau}\tilde{f}_o(\omega)d\omega$ where $\tilde{f}_o(-\omega) = \tilde{f}_o^*(\omega)$. Taking the unperturbed arrival time as the origin, the received signal will then be $f_r(\tau) = \int e^{-i\omega\tau}\mathcal{E}(\omega)\tilde{f}_o(\omega)d\omega$. The signal $f_r(\tau)$ is a Gaussian random variable whose complete statistics are determined by the covariance of $\tilde{f}_o\mathcal{E}$. Assuming[19] that variations in $\mathcal{E}_o(\omega)\tilde{f}_o(\omega)$ over a frequency

corresponding to the width of $\langle \mathcal{S}^*(-\frac{1}{2}\omega)\mathcal{S}(\frac{1}{2}\omega)\rangle$ can be neglected, this covariance is

$$\langle \tilde{f}_o^*(\omega')\mathcal{S}^*(\omega')\tilde{f}_o(\omega)\mathcal{S}(\omega)\rangle = \left|\tilde{f}_o\left(\tfrac{1}{2}(\omega + \omega')\right)\right|^2 \Lambda(\omega - \omega') \tag{4.6}$$

with

$$\Lambda(\omega) = \frac{\left(i\frac{\omega}{\omega_1}\right)^{\frac{1}{2}}}{\sin\left(i\frac{\omega}{\omega_1}\right)^{\frac{1}{2}}}$$

where the small α limit of Eq. (1.19) has been used and $\omega_1 = \omega_o\alpha/6 = c_g L^2/R^2\hat{\rho}(0,0)$.

Denoting the received intensity $f_r^2(\tau)$ by $\mathcal{I}(\tau)$, the average $\langle\mathcal{I}(\tau)\rangle$ is a measure of the distribution of energy over arrival times. According to Eq. (4.6) it is

$$\frac{\langle \mathcal{I}(\tau)\rangle}{\left\langle \int_{-\infty}^{\infty} \mathcal{I}(\tau)d\tau \right\rangle} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,\Lambda(\omega)d\omega = \begin{bmatrix} 0 & \tau < 0 \\ -2\omega_1 \sum_{n=1}^{\infty} (-1)^n (n\pi)^2\, e^{-n^2\pi^2\omega_1\tau} & \tau > 0 \end{bmatrix} \tag{4.7}$$

and vanishes for $\tau < 0$ because the integrand is analytic in the upper half plane. Evidently, all the energy comes in after the unperturbed arrival time and is confined to a region $0 \lesssim \tau \lesssim (\omega_1\pi^2)^{-1}$. The net retardation is consistent with what was said above about the phase of $\langle \mathcal{S}^*(\omega')\mathcal{S}(\omega)\rangle$. The complete absence of energy for $\tau < 0$ is peculiar to the limit of small α and will later be seen to have a simple physical interpretation (see Sec. (6)).

For a sharp transmitted pulse the distribution of energy over arrival times can be thought of as being due to two effects. One is the wander in arrival time of the center of the pulse and the other is spreading of the

pulse around its center. The two effects are in principle distinct. For the simple case of propagation in an unsaturated regime where $\mathcal{S}(\omega) = e^{i\omega\hat{\tau}}$, the wander is of order $\langle\hat{\tau}^2\rangle^{\frac{1}{2}}$ while the spread is just the width of the transmitted pulse. As we will now see, in the saturated regime the spread and wander are roughly equal. The width of $\langle\mathcal{J}(\tau)\rangle$ measures the sum of spread and wander. A quantity which measures the spreading, independent of wander, is

$$P(\tau) = \frac{\langle\int_{-\infty}^{\infty}\mathcal{J}(\tau + \tau')\mathcal{J}(\tau')d\tau'\rangle}{\left(\int_{-\infty}^{\infty}\langle\mathcal{J}(\tau')\rangle d\tau'\right)^2} \tag{4.8}$$

When $\tilde{f}_o\mathcal{S}$ has a Gaussian distribution $P(\tau)$ is

$$P(\tau) = \frac{P_o(\tau)}{2\pi}\int_{-\infty}^{\infty}|\Lambda(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\tau}|\Lambda(\omega)|^2 d\omega \tag{4.9}$$

where

$$P_o(\tau) = \frac{\left(\int_{-\infty}^{\infty}f_o(\tau + \tau')f_o(\tau')d\tau'\right)^2}{\left(\int_{-\infty}^{\infty}f_o^2(\tau')d\tau'\right)^2} \tag{4.10}$$

The two terms in $P(\tau)$ have the same height at $\tau = 0$. The spike proportional to $P_o(\tau)$ falls rapidly leaving the second term whose width is a measure of the spread. Comparing with Eq. (4.7) one sees that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\tau}|\Lambda(\omega)|^2 d\omega = \frac{\int_{-\infty}^{\infty}\langle\mathcal{J}(\tau + \tau')\rangle\langle\mathcal{J}(\tau')\rangle d\tau'}{\left(\int_{-\infty}^{\infty}\langle\mathcal{J}(\tau')\rangle d\tau'\right)^2} \tag{4.11}$$

and it is clear that the spread and wander are essentially the same. A physical interpretation of the two pieces of $P(\tau)$ will be given in Sec. (5). Finally, a useful formula is

$$|\Lambda(\omega)|^2 = \frac{|\omega/\omega_1|}{\sin^2\left(\frac{\omega}{2\omega_1}\right)^{\frac{1}{2}} + \sinh^2\left(\frac{\omega}{2\omega_1}\right)^{\frac{1}{2}}} \tag{4.12}$$

It is interesting to ask why it is that the square of the autocorrelation of f_o rather than the autocorrelation of f_o^2 appears in P_o. The answer is that when $\tilde{f}_o(\omega)\mathcal{S}(\omega)$ has a Gaussian distribution, the medium cannot transmit any information that is not contained in the coherence $\langle \tilde{f}_o^*(\omega')\mathcal{S}^*(\omega')\tilde{f}_o(\omega)\mathcal{S}(\omega)\rangle$. As given by Eq. (4.6) this coherence depends only on $|\tilde{f}(\omega)|^2$ and the medium can only transmit information about the autocorrelation of f.

The statistics of the signal as a function of spacial wave numbers can be analyzed in a similar way. Multiplying $\mathcal{S}(\vec{r},\vec{r}_o,t)$ by a suitable function of $\vec{r}_o$ and integrating over $\vec{r}_o$ one can represent a boundary condition at $z = 0$ corresponding to, say, a plane wave emerging from a finite aperture. For such a signal, the Fourier transform

$$\tilde{\mathcal{S}}_o(\vec{\ell}) = \frac{1}{(2\pi)^2}\int e^{-i\vec{r}\cdot\vec{\ell}}\mathcal{S}_o(\vec{r})d^2r \tag{4.13}$$

will be sharply peaked around some $\vec{\ell} = \vec{\ell}_o$. With the correspondences $t \to \vec{\ell}$, $\omega \to \vec{r}$ one can proceed as above and discuss spread and wander in $\vec{\ell}$. Again the medium can only transmit information contained in $\langle \mathcal{S}^*(\vec{r})\mathcal{S}(\vec{r}')\rangle$ which will typically depend only on $|\mathcal{S}_o(\frac{1}{2}(\vec{r} + \vec{r}'))|^2$.

5. CORRECTION TO THE $\alpha = 0$ LIMIT

The leading corrections to the $\alpha = 0$ limit are computed in Appendix B. The main results are as follows.

The order α correction to $\langle I^n\rangle$ is dominated by fluctuations near the transmitter and receiver. This is not unexpected since near their endpoints the paths cannot be separated into uncorrelated pairs. Explicitly $\langle I^n\rangle$ is

$$\langle I^n\rangle = n!\langle I\rangle^n\left(1 + \tfrac{1}{2}n(n-1)\alpha C + O(\alpha^2)\right) \tag{5.1}$$

where

$$C = \frac{(3\pi)^{\frac{1}{2}}}{4}\ \frac{L\int_0^\infty q^2\tilde{\rho}(q,0)dq}{\int_0^\infty q\tilde{\rho}(q,0)dq} \tag{5.2}$$

and $\tilde{\rho}$ is the three-dimensional Fourier transform of ρ

$$\rho(|\vec{x}|,t) = \frac{4\pi}{|\vec{x}|}\int_0^\infty q\ \sin(q|\vec{x}|)\tilde{\rho}(q,t)dq \tag{5.3}$$

The consequences of the fact that the error grows with n were noted in the Introduction. Note that the correction to $\langle I^n\rangle$ is positive. This means that the intensity fluctuations overshoot (i.e., become larger than Rayleigh) near the boundaries of the saturated regime.

The correction to a general correlation can also be computed. They are always fractionally small. For example, $\langle\mathcal{E}^*(2)\mathcal{E}^*(2)\mathcal{E}(1)\mathcal{E}(1)\rangle$ is proportional $e^{-D(1,2)}$ in the $\alpha = 0$ limit and the correction to it is of order $\alpha e^{-D(1,2)}$. The corrections to intensity correlations are the most interesting. In the $\alpha = 0$ limit $\langle I(t_1)I(t_2)\rangle$ is equal to $\langle I\rangle^2\left[1 + e^{-D(t_1 - t_2)}\right]$. At $t_1 = t_2$ the order α correction is given by Eq. (5.1). However, at large $|t_1 - t_2|$, $\langle I(t_1)I(t_2)\rangle$ must approach $\langle I\rangle^2$ and the corrections must go to zero. It turns out that half the correction dies like $e^{-D(t_1 - t_2)}$ but the other half falls much more slowly, leading to a coherence tail. (Note that this is consistent with what was said above about the corrections always being fractionally small.) For the general intensity correlation the coherence tail is

$$\langle I(\vec{r}_1,\vec{r}_{o1},t_1)I(\vec{r}_2,\vec{r}_{o2},t_2)\rangle = \tag{5.4}$$

$$\langle I\rangle^2\left[1 + e^{-D(1,2)} + \frac{\alpha\sqrt{3\pi}}{8}\ \frac{L\int_0^\infty q^2\tilde{\rho}(q,t_1 - t_2)\left[J_o(|\vec{r}_1 - \vec{r}_2|q) + J_o(|\vec{r}_{o1} - \vec{r}_{o2}|q)\right]dq}{\int_0^\infty q\tilde{\rho}(q,0)dq}\right]$$

where J_o is a Bessel function and specializing to $\vec{r}_1 = \vec{r}_2$ and $\vec{r}_{o1} = \vec{r}_{o2}$ produces

$$\langle I(t_1)I(t_2)\rangle = \langle I\rangle^2\left[1 + e^{-D(t_1 - t_2)} + \frac{\alpha\sqrt{3\pi}}{4}\,\frac{L\int_0^\infty q^2\tilde{\rho}(q,t_1 - t_2)dq}{\int_0^\infty q\tilde{\rho}(q,0)dq}\right] \tag{5.5}$$

Eq. (5.1) is not reproduced at $t_1 = t_2$ because a term of order $\alpha e^{-D(1,2)}$ has been dropped from both Equations (5.4) and (5.5).

6. FERMAT PATHS

There is an interesting connection between averages of the path integral and averages over Fermat paths which satisfy the perturbed ray equation

$$\vec{r}''(z) + \vec{\nabla}\mu(\vec{r}(z) + \vec{e}_z z) = 0 \tag{6.1}$$

where $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. This will be illustrated for the special case of sources and receivers located at $\vec{r} = \vec{r}_o = 0$ so that $\mathcal{S}$ is a function only of time. The path integral for $\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle$ is

$$\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle = \left\langle \frac{1}{4k^2}\int d^2(\text{paths})\exp\left[ik\int_0^R\left[\tfrac{1}{2}\left(\vec{r}_1{}'(z)\right)^2 - \tfrac{1}{2}\left(\vec{r}_2{}'(z)\right)^2 - \mu\left(\vec{r}_1(z) + \vec{e}_z z, 0\right) + \mu\left(\vec{r}_2(z) + \vec{e}_z z, t\right)\right]dz\right]\right\rangle \tag{6.2}$$

In the saturated region we know that for $\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle$ to be non-vanishing, t must be small and that only paths for which $|\vec{r}_1(z) - \vec{r}_2(z)|$ is small ($\sim L/\Phi$) contribute. Changing variables to $\vec{w}(z) = \frac{1}{2}\left(\vec{r}_1(z) + \vec{r}_2(z)\right)$ and $\vec{v}(z) = \vec{r}_1(z) - \vec{r}_2(z)$, the path integral can then be approximated by

$$\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle = \left\langle \frac{1}{4k^2}\int d^2(\text{paths})\ \exp\left[-ik\int_0^R \left[\vec{v}(z)\cdot\left\{\vec{w}''(z) + \vec{\nabla}\mu\left(\vec{w}(z) + \vec{e}_z z, 0\right)\right\}\right.\right.\right.$$
$$\left.\left.\left. - t\dot{\mu}\left(\vec{w}(z) + \vec{e}_z z, 0\right)\right] dz\right]\right\rangle \qquad (6.3)$$

where the first term in the argument of the exponential has been integrated by parts and a dot indicates differentiation with respect to time. The integration over the path $\vec{v}(z)$ produces a "δ-functional" which forces $\vec{w}'' + \vec{\nabla}\mu$ to vanish for all z. Thus the integral over $\vec{w}(z)$ is restricted to paths which satisfy the ray equation (6.1). This is a general feature of the saturated region. Higher order correlations are dominated by configurations where paths $\vec{r}_i(z)$ and $\vec{r}_{i'}(z)$ are pair-wise close. A similar analysis shows that for each such pair, the path $\vec{w}_i(z) = \frac{1}{2}\left(\vec{r}_i(z) + \vec{r}_{i'}(z)\right)$ satisfies Eq. (6.1).

Equation (6.3) can be further analyzed. For most media $\dot{\mu}$ and μ are statistically independent. The average, $\langle\rangle$, can then be thought of as two independent averages $\langle\ \rangle_\mu$ and $\langle\ \rangle_{\dot{\mu}}$ over μ and $\dot{\mu}$. The δ-function of $\vec{w}'' + \vec{\nabla}\mu$ that is produced by the integration over $\vec{v}$ is effected only by the average $\langle\ \rangle_\mu$ while the phase $\exp[ikt\int\dot{\mu}]$ is effected only by the other average $\langle\ \rangle_{\dot{\mu}}$. It is therefore possible to write $\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle$ as the integral over paths $\vec{w}$ of a μ-averaged δ-functional which can be interpreted as the probability that a given path $\vec{w}$ will satisfy Eq. (6.1) times a phase which is to be averaged over $\dot{\mu}$. To do this correctly, it is necessary to go back to the definition of the path integral in Eq. (2.1). The integration variables $\vec{v}_k$ and $\vec{w}_k$, $k = 1,2,\ldots,n$ are then discreet and the mathematics is straightforward. The integration over the $\vec{v}$'s can be done trivially and after some manipulation, one finds

$$\langle \mathcal{S}^*(t)\mathcal{S}(0)\rangle = \int d(\text{paths})'P(\text{path})\left\langle \exp\left[ikt\int_0^R \dot{\mu}\left(\vec{w}(z) + \vec{e}_z z, 0\right)dz\right]\right\rangle_{\dot{\mu}} \qquad (6.4)$$

where the integration d(paths)$'$ is over paths $\vec{w}(z)$ with a modified volume element

$$d(\text{paths})' = \frac{1}{(4\pi)^2}\left(\prod_{j=1}^{n} d^2 w_j\right)\left(\frac{n}{R}\right)^{2n} \qquad (6.5)$$

which does not contain k and

$$P(\text{path}) = \left(\frac{n}{R}\right)^{2n-2}\left\langle \prod_{j=1}^{n-1} \delta^2\left(\frac{n^2}{R^2}(\vec{w}_{j+1} + \vec{w}_{j-1} - 2\vec{w}_j) + \vec{\nabla}\mu(\vec{w}_j + \vec{e}_z z_j, 0)\right)\right\rangle_\mu \qquad (6.6)$$

is the probability that $\vec{w}$ will satisfy the finite difference approximation

$$\frac{n^2}{R^2}(\vec{w}_{j+1} + \vec{w}_{j-1} - 2\vec{w}_j) + \vec{\nabla}\mu(\vec{w}_j + \vec{e}_z z_j, 0) = 0 \qquad (6.7)$$

to the ray equation. In the limit $n \to \infty$, P(path) is the probability (with a measure d(paths)$'$) that $\vec{w}$ will satisfy Eq. (6.1). Equation (6.4) shows explicitly that $\langle \mathcal{E}^*(t)\mathcal{E}(0)\rangle$ is a sum over Fermat paths with fluctuating phases $kt\int\dot{\mu}$. Finally, bringing the average of $\dot{\mu}$ inside the exponential yields

$$\langle \mathcal{E}^*(t)\mathcal{E}(0)\rangle = \int d(\text{paths})' P(\text{path}) \exp\left[\frac{-k^2t^2}{2}\left\langle\left(\int_0^R \dot{\mu}\left(\vec{w}(z) + \vec{e}_z z, 0\right) dz\right)^2\right\rangle_{\dot{\mu}}\right] \qquad (6.8)$$

In the Markov approximation where $\vec{w}(z)$ is neglected relative to $\vec{e}_z z$ in the average of $\dot{\mu}$, Eq. (6.8) becomes

$$\langle \mathcal{E}^*(t)\mathcal{E}(0)\rangle = \exp\left[-\tfrac{1}{2}\phi^2\left(\frac{t}{T}\right)^2\right] \int d(\text{paths})' P(\text{path})$$

$$= \exp\left[-\tfrac{1}{2}\phi^2\left(\frac{t}{T}\right)^2\right]\langle I\rangle \qquad (6.9)$$

which is the standard result.

This provides a new way to look at the Markov approximation. It requires that an average like $\left\langle\left(\int_0^R \dot{\mu}\left(\vec{w}(z) + \vec{e}_z z, 0\right) dz\right)^2\right\rangle$ along a path which

satisfies the perturbed ray equation (6.1) should be well approximated by the corresponding average $\langle \left(\int_0^R \mu(\vec{e}_z z, 0) dz \right)^2 \rangle$ along the unperturbed ray. For a homogeneous and isotropic medium, this will be the case as long as the rms multiple scattering angle $(\langle\mu^2\rangle R/L)^{\frac{1}{2}}$ is small.

According to Eq. (6.8), $\langle \mathcal{E}^*(t)\mathcal{E}(0)\rangle$ can in principle be computed by a geometric optics method which searches out the rays which satisfy the perturbed ray equation. Geometric optics corresponds to an approximate evaluation of the path integral by the method of stationary phase.[6] In the saturated region the stationary phase approximation will in fact be valid since for $\Phi > 1$ the phase $k\int_0^R \mu dz$ is necessarily large. To get Gaussian statistics for $\mathcal{E}$, it is necessary that there be several rays connecting a given source and receiver. In path integral language this means that there will be multiple stationary phase points and $\mathcal{E}$ will be a discreet sum $\sum_k A_k e^{i\varphi_k}$ over contributions, one from each stationary phase point or ray. The phases φ_k and amplitudes A_k as well as the number of rays will fluctuate with μ yielding Gaussian statistics for $\mathcal{E}$.

It is difficult to prove rigorously that there are always multiple rays in the saturated regime. However there is a simple construction which shows the essential physics. At one fixed time the rays are stationary points of the path length S defined by

$$S = k\int_0^R \left[\tfrac{1}{2}(\vec{r}\,'(z))^2 - \mu\,(\vec{r}(z) + \vec{e}_z z)\right] dz \quad . \tag{6.10}$$

Let $S(\vec{r})$ be S evaluated for the special paths that go in a straight line from the source at $(\vec{0},0)$ to an arbitrary point $(\vec{r}, z_0)$ with $0 < z_0 < R$ and then follow another straight line from (r,z_0) to the receiver at $(\vec{0}, R)$. Multiple stationary points of $S(\vec{r})$ as a function of $\vec{r}$ will be indicative of multiple stationary points of the complete functional in Eq. (6.10) and the spacing of such points in $\vec{r}$ will be similar to the

spacing between multiple rays. Now doing a simple integral shows that $S(\vec{r})$ can be written as

$$S(\vec{r}) = \tfrac{1}{2} (\vec{r})^2 B - S_1(\vec{r}) \tag{6.11}$$

where $B = \dfrac{kR}{z_0(R-z_0)}$ and S_1 is $k\mu$ integrated along the above mentioned path. To simplify $S(\vec{r})$, B^{-1} can be replaced by its average value $R/(6K)$. Then defining $\vec{r} = L\,\vec{u}$ and $S_1(\vec{r}) = \Phi\, f(\vec{u})$ the quantity to be studied is

$$\tfrac{1}{2}\,\Omega\,\vec{u}^{\,2} - \Phi f(\vec{u}) \tag{6.12}$$

and we are interested in its stationary points which satisfy

$$\Omega\,\vec{u} - \Phi\vec{\nabla}\, f(\vec{u}) = 0 \quad . \tag{6.13}$$

By construction f is a random function of order unity which changes by order one when its argument changes by order one; i.e. $|\vec{\nabla} f| \sim 1$ and $\vec{\nabla} f$ changes sign roughly each unit in $\vec{u}$. For $\Omega \gg \Phi$ the first term in Eq. (6.13) dominates and there will be a single solution near $\vec{u} = 0$. This is the unsaturated regime. In the saturated regime, $\Phi \gg \Omega$, the random character of f guarantees that there will generally be many solutions, spaced by about one unit in $\vec{u}$ (a distance L in $\vec{r}$) and filling up the interval $0 < |\vec{u}| < \Phi/\Omega$ $(0 < |\vec{r}| < \Phi L/\Omega)$. To find the other boundary of the saturated regime, $\Phi > 1$, we have to ask when the multiple rays are physically meaningful. From their interpretation as stationary phase points of the path integral it can be verified that two rays will be physically distinct if S varies by a quarter cycle, i.e., order unity between the two. The variation in S between two solutions of Eq. (6.13) will be roughly Φ and if they are to represent physically distinct rays Φ must be greater than unity.

An experiment with a pulsed source will tend to see several arrivals corresponding to the multiple Fermat paths. This random multipathing is the origin of the rapid fall-off of frequency coherence which takes

place in the saturated regime. To see how the orders of magnitude work, the difference in travel time between the ray nearest $\vec{u} = \vec{0}$ and the furthest one out at $|\vec{u}| \sim \Phi/\Omega$ is $t_0 = \omega^{-1}(\Phi^2/2\Omega \pm \Phi) \approx \Phi^2/(2\Omega\omega)$ where the two terms come from the two terms in Eq. (6.12) and it has been assumed that $\Phi >> \Omega$. Frequencies which differ by more than t_0^{-1} will then be incoherent, in agreement with Eqs. (1.19) and (1.21). Note that t_0 is positive. This is why in the limit $\alpha = 0$ all the energy arrives after the unperturbed arrival time and $\langle \mathcal{I}(\tau) \rangle$ vanishes for $\tau < 0$. Also the two terms in $P(\tau)$ (Sec. 4) can easily be interpreted in terms of fluctuating multipath. The spike $P_0(\tau)$ is the autocorrelation of each arrival with itself and the broad second term is the autocorrelation of different arrivals. Finally a word of caution. The above construction vastly underestimates the number of rays. In reality the number of rays is probably an exponential of Φ/Ω rather than Φ/Ω as the construction would imply. It may be extremely difficult to actually resolve the arrivals.

It is interesting to consider the transition into the saturated regime in terms of propagation of a pulse. Consider first crossing the line $\Phi = \Omega$ from the region where both Φ and Ω are large but $\Omega < \Phi$. With Φ and Ω large but well outside the saturated region, one knows from the Rytov approximation that the receiver will see a single arrival with a considerable wander in time of arrival. At the boundary of the saturated region the pulse will begin to split into several arrivals and well inside the saturated region there will be many arrivals that are spread out over a time long compared to the original wander in the single pulse. Crossing the boundary $\Phi = 1$ from the region where both Φ and Ω are small is rather different. In this case one knows that well outside the saturated region, there will be a single arrival with no discernible wander in time of arrival accompanied by a small scattered wave spread over a continuum of arrival times. As the boundary of the saturated region is approched, the single peak will shrink and the scattered wave will grow in amplitude. Well inside the saturated region, the original peak will have disappeared completely and the now large scattered wave will have broken up into a number of discrete arrivals.

7. Media with Multiple Scales

So far it has been assumed that the fluctuations in μ can be characterized by a single scale size L. Technically, this requires that the expansion of $\hat{\rho}$,

$$k^2 R\hat{\rho}(|\vec{r}|,0) = \Phi^2 \left(1 - \frac{\vec{r}^{\,2}}{2L^2} + a\,\frac{\vec{r}^{\,4}}{4L^4} + \ldots.\right) \tag{7.1}$$

through order $\vec{r}^{\,4}$ exists and that the coefficient a is of order unity. There are cases of practical importance where this is not true. For example, optical index of refraction fluctuations induced by Kolomogorov turbulence have the property that the (three dimensional) Fourier transform $\tilde{\rho}(q)$ of ρ behaves like $|q|^{-11/3}$ over a long interval in q and the expansion in Eq.(7.1) makes sense only when the cutoff (inner scale) is taken into account and then a is very large. This and the following section are devoted to these media with multiple scales. It will be assumed that $\tilde{\rho}(q)$ goes like $|q|^{-2-p}$ for large q where $4 > p > 1$. (If p is greater than four the medium acts like one with a single scale size and for $p < 1$ it is so singular that $\langle\mu^2\rangle$ does not exist.) In practice there is always some physical cutoff at large q (inner scale). However, the effects of such a cutoff will be ignored in what follows.

For $p > 2$, the length parameter L will be defined by Eq.(1.13) as before and in the case $p < 2$, L will be defined by

$$\hat{\rho}(|\vec{r}|,0) = \hat{\rho}(0,0)\left[1 - \frac{1}{2}\left|\frac{\vec{r}}{L}\right|^p\right] \tag{7.2}$$

for small $|\vec{r}|$. For Kolomogorov turbulence, p is equal to 5/3 and $\hat{\rho}(0,0)$ and L are related to Tatarskii's C_n by $2.91C_n^2 = \hat{\rho}(0,0)L^{-5/3}$. The parameters Φ and Ω continue to be defined by Eqs. (1.6) and (1.7).

The main qualitative difference between propagation in single and multiple scale media is that in the latter case there is more than one saturated regime. In terms of the Fermat paths of the last section, it turns out that in a multiple scale medium the smaller scale inhomogenieties can make multiple Fermat paths before the large ones do. This leads to a new kind of saturated regime. Even in a single scale medium with $p > 4$ the line $\Phi = \Omega$ is not a sharp boundary. In reality there is a transition zone where random focusing along single Fermat paths produces intensity fluctuations bigger than Rayleigh. As p decreases below four this transition zone opens up and becomes a new saturated regime. The boundaries of this new regime can be found by studying the object $\frac{1}{2}\,\Omega\vec{u}^2 - \Phi f(\vec{u})$ of Eq. (6.12).

To see when the smaller scales can make multiple Fermat paths, imagine throwing out all scale sizes larger than λL where $1 > \lambda > 0$. The new scale length will be λL and Φ and Ω will be replaced by $\lambda^{p/2}\Phi$ and $\lambda^2\Omega$. The combination Φ/Ω becomes $\lambda^{(p-4)/2}\Phi/\Omega$ and is equal to unity when $\lambda = (\Phi/\Omega)^{2/(4-p)}$. Thus if $p < 4$ the small scales can make multiple Fermat paths when $\Phi < \Omega$, i.e., before the large ones do at $\Phi = \Omega$. However, if these multiple paths are to be physically meaningful $\lambda^{p/2}\Phi$ must be greater than unity and the smallest permissible value of λ is $\Phi^{-2/p}$. Putting everything together, the small scales can make meaningful multiple Fermat

paths when $\Phi^{4/p}/\Omega > 1$. This is one boundary of the new saturated regime. To find the other boundary, we need to ask when the multiple Fermat paths can be separated by L. For a given λ the minima of $\frac{1}{2}\lambda^2\Omega\vec{u}^2 - \lambda^{p/2}\Phi f(\vec{u})$ extend out to a maximum $|\vec{u}|$ which is the largest value of $|\vec{u}|$ for which the equation $\lambda^2\Omega\vec{u} - \lambda^{p/2}\Phi\vec{\nabla}f(\vec{u})$ can be solved. The maximum $|\vec{u}|$ is $(\Phi/\Omega)\lambda^{(p-4)/2}$ and noting that $\vec{u}$ is distance in units of λL one sees that the Fermat paths can be separated by L when $(\Phi/\Omega)\lambda^{(p-2)/2} = 1$. For $p > 2$ the most separated paths are due to large scales with $\lambda = 1$ and the other boundary of the new region is $\Phi = \Omega$. However, if $p < 2$ the smaller scales produce the largest separation and taking the smallest permissible value $\Phi^{-2/p}$ for λ one sees that there can be Fermat paths separated by L when $\Phi^{2/p}/\Omega > 1$. The regime where there are meaningful multiple Fermat paths all lying within L of each other will be called the <u>partially saturated regime</u>. The regime where the spacing between Fermat paths can be greater than L is analagous to the saturated regime of the single scale case and will be called the <u>fully saturated regime</u>. The boundaries of these regimes are summarized in Table 1.

	Partially Saturated Regime	Fully Saturated Regime
$2<p<4$	$\Phi>1, \Phi^{4/p}/\Omega>1, \Phi/\Omega<1$	$\Phi>1, \Phi/\Omega>1$
$1<p<2$	$\Phi>1, \Phi^{4/p}/\Omega>1, \Phi^{2/p}/\Omega<1$	$\Phi>1, \Phi^{2/p}/\Omega>1$

Table 1. Boundaries of the Saturated Regimes

Although these boundaries have been obtained with a hueristic Fermat path argument they are in agreement with what one finds from more precise calculations. It is known that outside the saturated regimes the intensity fluctuations $(\langle I^2\rangle - \langle I\rangle^2)/\langle I\rangle^2$ are small, implying both the validity of the Rytov approximation and the absence of saturation. Inside the saturated regimes (as given by Table 1) the intensity fluctuations as computed in the Rytov approximation are large, signaling the onset of saturation. The line between the fully and partially saturated regimes corresponds to the place where two pairs of paths, in the sense of Sec. (3), can be separated by more than L. When they are separated by more than L the pairs of paths are completely independent (full saturation) and Gaussian statistics for $\mathcal{E}$ follows immediately. If all pairs are within L of each other (partial saturation) then one expects that at least some statistics will not be Gaussian.

Nothing that was done in Sec. (2) or App. (A) depended in any essential way on the assumption of a single scale. The reader can verify that Eq. (1.14) for $\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle$ at equal frequencies continues to hold whenever the parabolic wave equation is valid. The only subtle point is that for $p < 2$ the rms scattering angle is not well defined and, correspondingly, in App. A, Eq. (A.6) cannot be approximated by Eq. (A.9). However, a rather straightforward analysis of Eq. (A.6) shows that the fractional error in Eq. (1.14) is of order $D(k^{-1},0)$ and it is known that $D(k^{-1},0) < 1$ is the validity condition for the parabolic wave equation when

$p < 2$. Turning to coherences in frequency, there is however a significant defect in the theory if $p < 2$. When p is less than 2, the path integral in Eq.(2.22) cannot be approximated by that in Eq.(2.24) and Λ must be understood as a function defined by Eq.(2.22) whose evaluation would require a numerical calculation.

In the fully saturated regime where pairs of paths can be separated by L or greater, the arguments of Sec. (3) proceed as before. One readily verifies that in the fully saturated regime the statistics of $\mathcal{E}$ are Gaussian and the discussion of Sec. (4) applies (except Eq.(4.7) which assumes Eq.(1.21) for Λ). Eqs.(B.12) and (B.17) of App. B hold in the multiple scale case. The reader can then verify that for $p > 2$, Eqs.(5.1),(5.2),(5.4) and (5.5) *for the corrections* to Gaussian statistics continue to hold in the fully saturated regime and that for $p < 2$ these same equations hold if α is replaced by α' where

$$\alpha' = \frac{4(p+1)^{3/p}\Gamma(3/p)}{3^{3/2}\pi^{1/2}p} \frac{\Omega}{\Phi^{(6-2p)/p}} \tag{7.3}$$

The situation for the fully saturated regime is summarized in Table 2.

	Boundaries	Limiting Statistics	Corrections to the Limiting Statistics
$2<p<4$	Unchanged	Unchanged	Unchanged
$1<p<2$	Replace $\Phi/\Omega>1$ by $\Phi^{2/p}/\Omega>1$	Unchanged except that Λ is not known explicitly	Replace α by α'

Table 2. Changes Needed to Apply the Formulas of Secs.(1)-(5) to the Fully Saturated Regime.

The higher order statistics in the partially saturated regime are more complicated. For the case $p < 2$ everything can be worked out in detail and the results will be given in the next section. However, for $p > 2$ the path integrals yield only qualitative information: It is summarized in Appendix D.

Finally, in multiple scale media the notion of multiple Fermat paths should be used with care. They exist but there are so many of them that they cannot, even in principle, be completely resolved. Nevertheless, the notion is useful in interpreting the path integral calculations and will continue to be employed.

8. The Partially Saturated Regime for $p < 2$

The partially saturated regime for $p < 2$ is of considerable practical importance. Many atmospheric optics experiments lie in this region and, luckily, the complete statistics of $\mathcal{E}$ can be worked out. There is a natural small parameter β defined by

$$\beta = \left(\Omega/\Phi^{4/p}\right)^{2-p} \tag{8.1}$$

For $p = 5/3$, β is related to Tatarskii's[1] C_n by $\beta = 1.19\, C_n^{-4/5} R^{-11/15} k^{-7/15}$ and to the intensity fluctuations as computed in the Rytov approximation by $\left.\langle(\ln I)^2\rangle - \langle \ln I\rangle^2\right|_{\text{Rytov}} = 0.80\beta^{-5/2}$.

The signal statistics will be given through order β.

Partial saturation is due to the appearance of multiple Fermat paths all lying within L of each other. The larger scales (~L) will tend to correlate the locations of these paths leading in general to a complicated statistics. However, for $p < 2$ the spectrum is so heavily weighted toward small scales that the locations of the Fermat paths turn out to be uncorrelated. This is not the case for $p > 2$ where the multiple Fermat paths become correlated and the path integral yields only qualitative information (see App.D). Even for $p < 2$ where the locations of the paths are uncorrelated the large scales can still correlate the phases along different Fermat paths. We will see this at the end of the section when coherences in frequency are studied.

Consider Eqs.(3.1) and (3.2) for $\langle I^2\rangle$ in the partially saturated regime with $p < 2$. In the integration region (a) the separation between members of a pair of paths $\vec{v}_1(z)$ (using the

notation of App. B) must be such that $d(|\vec{v}_1(z)|) \lesssim 1$, i.e., $|\vec{v}_1(z)| \lesssim L/\Phi^{2/p}$. The distance $\vec{v}_2(z)$ between pairs (again in the notation of App.B) will be limited by the oscillating terms in the path integral to values such that $\Omega|\vec{v}_2(z)||\vec{v}_1(z)| \sim L^2$ or $|\vec{v}_2(z)| \lesssim L\Phi^{2/p}/\Omega$. Note that the ratio of the cutoff on $|\vec{v}_2|$ to that on $|\vec{v}_1|$ is $\Phi^{4/p}/\Omega$ and is large. Now both $|\vec{v}_1|$ and $|\vec{v}_2|$ are small compared to L and Eq.(7.2) can be used to evaluate M in Eqs.(3.2) or (B.3). Taking account of the fact that $|\vec{v}_1| \ll |\vec{v}_2|$, the expression for M in Eq.(B.3) of App.(B) then becomes[20]

$$M = \Phi^2 R^{-1} \int_0^R \left|\frac{\vec{v}_1(z)}{L}\right|^p dz - \Phi^2 R^{-1} p(p-1) \int_0^R \left|\frac{\vec{v}_1(z)}{L}\right|^2 \left|\frac{\vec{v}_2(z)}{L}\right|^{p-2} dz \qquad (8.2)$$

and when $|\vec{v}_1/L| \sim \Phi^{-2/p}$ and $|\vec{v}_2/L| \sim \Phi^{2/p}/\Omega$ the second term on the right hand side of Eq.(8.2) is of order β and can be dropped. This is the same thing as saying that different pairs of paths in the path integral, or equivalently different Fermat paths in the sense of Sec.(6), are not correlated and $\langle I^2\rangle$ becomes $2\langle I\rangle^2$. What is happening is that for $p < 2$ the fractional power behavior of d at small separations is making the arguments of Sec.(3) valid even though the different pairs are separated by less than L. Note that this will only happen for $p < 2$. The generalization to $\langle I^n\rangle$ is straightforward and the result is a Rayleigh distribution with $\langle I^n\rangle = n!\,\langle I\rangle^n$.

The true test of the method comes when one evaluates the corrections to Rayleigh statistics. It is shown in App.C that to order β

$$\langle I^n\rangle = n!\langle I\rangle^n(1 + \tfrac{1}{2}\, n(n-1)C(p)\beta) \qquad (8.3)$$

where $C(p)$ is a constant which depends only on p. This constant is evaluated in App.(B) and $C(5/3) = 1.06$. The corrections are small for small β showing that the approximation scheme is consistent but there will be significant deviations from a Rayleigh distribution when $I/\langle I\rangle \gtrsim \sqrt{2/\beta C(p)}$.

The statistics of $\mathcal{E}(\vec{r},\vec{r}_o)$ as a function of source and receiver locations can be investigated in a similar way. One finds that they are Gaussian and at equal times and frequencies the results of Secs.(3) and (4) hold in the limit $\beta = 0$. There are coherence tails of order β. These are discussed in App.C.

In the fully saturated regime the dynamics of the medium enters only through $D(t)$. This is not always true in the partially saturated regime. It is true when the Taylor hypothesis is valid (a frozen field convected by a "wind") and the statistics in time can be obtained from the spatial statistics. However, one can consider a different kind of medium where the time dependence of μ is associated with linear waves whose dispersion relation is $\omega \sim k^{\delta/2}$. The Fourier transform of the second time derivative $\ddot{\rho}$ of ρ will then behave like

$$\tilde{\ddot{\rho}}(|\vec{q}|) = (\text{const})\,|\vec{q}|^{-(2+p-\delta)} \tag{8.4}$$

at large $|\vec{q}|$. For the Taylor hypothesis Eq.(8.4) holds with $\delta = 2$ and in general δ can be considered as being defined by Eq.(8.4). Assuming $p < 2$, the statistics of $\mathcal{E}$ at unequal times are Gaussian in the partially saturated regime provided that $p - \delta < 0$. This can be verified by explicitly computing the

corrections. For $p < 2$ and $p - \delta < 0$ the corrections to Gaussian statistics are fractionally small for small β and the results of Secs.(3) and (4) continue to hold at unequal times. However for $p - \delta > 0$, a direct calculation shows that the corrections to Gaussian statistics are <u>not</u> fractionally small and therefore that the approximation scheme of Sec.(3) is not consistent at unequal times.

To see what is happening for $p - \delta > 0$ one can compare the path integrals for $\langle I(t')I(t)\rangle$ and $\left\langle\left(\mathcal{E}^*(t')\right)^2\left(\mathcal{E}(t)\right)^2\right\rangle$. The latter is sensitive to the time dependence of the phase of $\mathcal{E}$ while the former is not. A rather involved but straightforward calculation then shows that for $p - \delta > 0$ the signal moves more rapidly in phase than in amplitude. This is to be contrasted with the case $p - \delta < 0$ where the time statistics are Gaussian and according to Eq.(1.18) there is no tendency to move in phase as opposed to amplitude. As long as $p < 2$ the signal has a Rayleigh distribution and over a long time the track of the signal will fill out a disc in the complex plane. The difference between $p - \delta < 0$ and $p - \delta > 0$ comes in how this disc is filled up. For $p - \delta < 0$ the signal is Gaussian and it will make a track of the type shown in Fig.(5a) which looks something like a random walk. However, for $p - \delta > 0$ the track will wrap around in phase and slowly move in and out in amplitude as shown in Fig. (5b).

These peculiar features of time statistics in the partially saturated regime can be understood in terms of Fermat paths. We know that $\mathcal{E}(t)$ is schematically $\sum_k A_k(t)e^{i\phi_k(t)}$ where the locations

of the paths are uncorrelated (for $p < 2$) but the large scales may correlate the phases $\phi_k(t)$. The question of random walking vs phase wrapping is equivalent to the question of whether or not the time derivatives $\frac{d}{dt}\phi_k = \dot{\phi}_k$ are correlated. For $p - \delta < 0$, the time derivatives are sufficiently weighted towards small scales that the $\dot{\phi}_k$ are uncorrelated and the signal random walks. However for $p - \delta > 0$, the effect of the large scales is strong enough to produce a correlated phase derivative common to all the Fermat paths.

Propagation of sound in the ocean is an example of a situation where $\mathcal{E}(t)$ phase wraps in the partially saturated regime.[7] For the ocean $p \approx 2$, $\delta \approx 0$ and in this special case it is possible to work out the detailed statistics of $\mathcal{E}(t)$.[7] However, for other combinations of p and δ it is not possible to compute fourth and higher moments of $\mathcal{E}(t)$ analytically, except when $p - \delta < 0$.

Checking consistency, it was stated above that for $p < 2$ the statistics of $\mathcal{E}(\vec{r},\vec{r}_o)$ as a function of $\vec{r}_o$ and $\vec{r}$ are Gaussian in the partially saturated regime. If the time derivatives on the right hand side of Eq.(8.4) were replaced by spatial derivatives we would have $\delta = 2$. Since $p - 2 < 0$ for $p < 2$ it is consistent that the statistics in $\vec{r}_o$ and $\vec{r}$ are Gaussian and that the statistics in time are Gaussian when the Taylor hypothesis (implying $\delta = 2$) is valid.

For $p < 2$ and $p - \delta < 0$, the statistics of $\mathcal{E}(\vec{r},\vec{r}_o,t)$ in the partially saturated regime are essentially the same as in the fully saturated regime. The reader may therefore wonder what the basic distinction between the regimes is. The answer turns out to lie in the statistics in frequency.

Let us examine the path integral for $\langle \mathcal{E}^*(\omega_1)\mathcal{E}(\omega_2)\mathcal{E}^*(\omega_3)\mathcal{E}(\omega_4)\rangle$. Up to a normalization it is

$$\langle \mathcal{E}^*(\omega_1)\mathcal{E}(\omega_2)\mathcal{E}^*(\omega_3)\mathcal{E}(\omega_4)\rangle \sim$$

$$\int d^4(\text{paths}) \exp\left[\frac{1}{2}\sum_{j=1}^{4}(-1)^j\frac{\omega_j}{c}\int_0^R (\vec{r}_j{}'(z))^2 dz - N\right] \tag{8.5}$$

where with the Markov approximation, surpressing time t

$$N = \frac{1}{2}\sum_{i,j=1}^{4}(-1)^{i+j}\ \frac{\omega_i\omega_j}{c^2}\int_0^R \hat{\rho}(|\vec{r}_i(z)-\vec{r}_j(z)|)dz \tag{8.6}$$

and for simplicity the medium has been assumed to be nondispersive. There are the usual two important regions of path space (a) and (b). Let us concentrate on (a) where $|\vec{r}_1-\vec{r}_2| < L/\Phi^{2/p}$ and $|\vec{r}_3-\vec{r}_4| < L/\Phi^{2/p}$. First we will see how Gaussian statistics arise in the fully saturated regime and then see how the partially saturated case differs. In the fully saturated case typical values of, say, $|\vec{r}_1-\vec{r}_3|$ are large compared to L and $\hat{\rho}(|\vec{r}_1-\vec{r}_3|)$ can be set equal to zero. Ignoring correlations between the different pairs then yields

$$N \approx \frac{1}{2}\left(\sum_{i,j=1}^{2} + \sum_{i,j=3}^{4}\right)(-1)^{i+j}\ \frac{\omega_i\omega_j}{c^2}\int_0^R \hat{\rho}(|\vec{r}_i(z)-\vec{r}_j(z)|)dz \tag{8.7}$$

which is a sum of two terms one of which depends on ω_1 and ω_2 and the other on ω_3 and ω_4 and the result is Gaussian statistics. In the partially saturated case typical values of $|\vec{r}_1-\vec{r}_3|$ are small compared to L and $\hat{\rho}(|\vec{r}_1-\vec{r}_3|)$ is approximately equal to $\hat{\rho}(0)$. Now

we have to set correlations between the different pairs of paths equal to $\hat{\rho}(0)$ rather than zero and N becomes

$$N \approx \frac{1}{2}\left(\sum_{j=1}^{4}(-1)^{j}\frac{\omega_j}{c}\right)^2 R\hat{\rho}(0)$$

$$+\frac{1}{2}\left(\sum_{i,j=1}^{2}+\sum_{i,j=3}^{4}\right)(-1)^{i+j}\frac{\omega_i\omega_j}{c^2}\int_0^R\left[\hat{\rho}(|\vec{r}_i(z)-\vec{r}_j(z)|)-\hat{\rho}(0)\right]dz \tag{8.8}$$

The path integral again factors into a product of two double path integrals and is expressable in terms of Λ as defined in Eq.(2.22). Collecting the contribution from both regions (a) and (b) and supplying the correct normalization yields

$$\frac{\left\langle\mathcal{E}^*(\omega_1)\mathcal{E}(\omega_2)\mathcal{E}^*(\omega_3)\mathcal{E}(\omega_4)\right\rangle}{\mathcal{E}_o^*(\omega_1)\mathcal{E}_o(\omega_2)\mathcal{E}_o^*(\omega_3)\mathcal{E}_o(\omega_4)} =$$

$$\exp\left[-\frac{1}{2}\left(\sum_{j=1}^{4}(-1)^{j}\frac{\omega_j}{c}\right)^2 R\hat{\rho}(0)\right]\left[\Lambda(\omega_2-\omega_1)\Lambda(\omega_4-\omega_3)+\Lambda(\omega_2-\omega_3)\Lambda(\omega_4-\omega_1)\right]. \tag{8.9}$$

Because of the common exponential factor in front of the two terms on the right hand side this is not Gaussian statistics. What it corresponds to is an $\mathcal{E}$ of the form $\mathcal{E}(\omega)/\mathcal{E}_o(\omega) = e^{i\omega\psi}\chi(\omega)$ where ψ is a real Gaussian random variable with $\langle\psi\rangle = 0$, $\langle\psi^2\rangle = R\hat{\rho}(0)c^{-2}$ and $\chi(\omega)$ is an independent complex Gaussian random variable with zero mean and covariances $\left\langle\chi(\omega)\chi(\omega')\right\rangle = \left\langle\chi^*(\omega)\chi^*(\omega')\right\rangle = 0$ and $\left\langle\chi^*(\omega)\chi(\omega')\right\rangle = \Lambda(\omega'-\omega)$. It is straightforward to verify that this

ansatz does in fact yield the correct 2n-th moment of $\mathcal{E}(\omega)$ in the partially saturated regime. In particular the second moment

$$\frac{\langle \mathcal{E}^*(\omega)\mathcal{E}(\omega')\rangle}{\mathcal{E}_o^*(\omega)\mathcal{E}_o(\omega')} = \left\langle \exp\left[i(\omega'-\omega)\psi\right]\chi^*(\omega)\chi(\omega')\right\rangle$$

$$= \left\langle \exp\left[i(\omega'-\omega)\psi\right]\right\rangle \left\langle \chi^*(\omega)\chi(\omega')\right\rangle$$

$$= \exp\left[-\frac{1}{2}(\omega-\omega')^2 R\hat{\rho}(0)c^{-2}\right]\Lambda(\omega'-\omega) \qquad (8.10)$$

comes out right. For a dispersive medium $\langle\psi^2\rangle$ becomes ω_g^{-2} as in Eq. (1.20) and c_g rather than c appears in Λ.

Thus the fundamental distinction between the fully and partially saturated regimes is that in the former the statistics in frequency are Gaussian while in the latter they correspond to a phase times a Gaussian. Well inside the partially saturated regime ω_g is small compared to the width in ω of Λ. The phase $e^{i\omega\psi}$ then dominates the moments of $\mathcal{E}(\omega)$, except for correlations involving only $|\mathcal{E}(\omega)|^2$ where ψ cancels. As the boundary $\Phi^{2/p}/\Omega = 1$ of the fully saturated regime is approached the width of $\Lambda(\omega)$ becomes comparable to ω_g and upon passing into the fully saturated regime Λ dominates the moments and the signal becomes Gaussian. In the terminology of Sec. (4), for partial saturation the spread is small compared to the wander. In pulse propagation $e^{i\omega\psi}$ represents a quasi-deterministic wander which dominates $\langle \mathcal{J}(\tau)\rangle$. The phase $e^{i\omega\psi}$ cancels out in the integral (Eq. (4.8)) for $P(\tau)$ and the spreading of a pulse is proportional to the inverse width of Λ.

In terms of Fermat paths $\mathcal{E}(\omega) = \sum_k A_k(\omega) e^{i\phi_k(\omega)}$, the non-Gaussing statistics can be understood as follows. Each $\phi_k(\omega)$ can be written as $\omega\psi + \Delta\phi_k(\omega)$ where $\omega\psi$ is a common phase generated by the larger scales. The phase differences $\Delta\phi_k(\omega)$ are due to the small scales. They vary from path to path and are responsible for the Gaussian factor $\chi(\omega)$. Note that only correlations in frequency measure ψ directly. Correlations in space or time see only $\vec{\nabla}\psi$ or $\dot{\psi}$ which for $p < 2$ and $p - \delta < 0$ are dominated by small rather than large scales, leading ultimately to Gaussian statistics. The phase wrapping in time for $p - \delta > 0$ is a remnant of ψ.

The statistics of $\mathcal{E}$ in the partially saturated regime are summarized in Table 3.

Intensity Distribution	Variations in			
	space	time		frequency
		$p - \delta < 0$	$p - \delta > 0$	
Rayleigh	Gaussian	Gaussian	phase wrapping	phase times a Gaussian

Table 3. The Statistics of in the Partially Saturated Regime for $p < 2$.

The reader may be curious as to what happens at $p = 2$. The "small" parameter β is then equal to unity but according to App. C the coefficient $C(2)$ in Eq. (8.3) vanishes. A detailed investigation[7] then shows that the corrections to Rayleigh statistics in the partially saturated regime are of order $(\ln\Phi)^{-1}$. More generally, if $p = 2$ and $\ln\Phi$ is large the statistics given in Table 3 apply with errors of order $(\ln\Phi)^{-1}$. At $p = 2$ it is possible to compute Λ. It is given[7] by Eq.(1.21) with $\omega_o\alpha$ replaced

by $\omega_o \alpha (\ell n \Phi)^{-1}$. In general, a medium with $|p-2| \ell n \Phi < 1$ will act like one with $p = 2$.

As mentioned before the case of partial saturation for $p > 2$ is discussed in App. D.

9. INHOMOGENEOUS AND ANISOTROPIC MEDIA

In practice, random media are only locally homogeneous and the covariance

$$\rho(\vec{x}-\vec{x}', t-t'; \bar{\vec{x}}) = <\mu(\vec{x},t)\,\mu(\vec{x}',t')> - <\mu(\vec{x},t)><\mu(\vec{x}',t')> \tag{9.1}$$

depends on position $\bar{\vec{x}} \equiv \frac{1}{2}(\vec{x}+\vec{x}')$. It is always assumed that the variations of ρ in $\vec{x}-\vec{x}'$ are much more rapid than those in $\bar{\vec{x}}$ but over a long propagation path the dependence on $\bar{\vec{x}}$ cannot always be neglected. Also, in an inhomogeneous medium $<\mu(\vec{x})> = \mu_0(\vec{x})$ will generally not be a constant and consequently cannot be absorbed in the definition $c = \omega/k$. Finally, the medium can be statistically anisotropic so that ρ depends on the orientation of $\vec{x}-\vec{x}'$ as well as its magnitude.

To obtain tractable path integrals in an inhomogeneous medium we will have to approximate the path dependence of $\bar{\vec{x}}$ in ρ by evaluating $\bar{\vec{x}}$ along some central path which will turn out to be an unperturbed ray. From Sec. (3) we know that paths are separated by $L\Phi/\Omega$ (the precise definitions of L, Φ and Ω for inhomogeneous anisotropic media will be given below) and the problem will be tractable if

(i) for changes in $\bar{\vec{x}}$ of order $L\Phi/\Omega$ the corresponding variations in ρ can be neglected. It will also turn out to be necessary to expand μ_0 in powers of distances between paths and we will have to require that

(ii) $\mu_0(\vec{x})$ is slowly varying over distances of order $L\Phi/\Omega$.

The one other condition is that

(iii) the parabolic wave equation is a valid approximation.

When $\mu_0(\vec{x})$ is not a constant this requires that the normals to the wave fronts in the "unperturbed problem" where $\mu(\vec{x}) = \mu_0(\vec{x})$ remain close to the z-axis. If this is true locally but not globally, then solutions based on the parabolic approximation can be patched together in the obvious way.

When conditions (i), (ii) and (iii) are met it is reasonably straightforward to extend the path integral method to inhomogeneous and anisotropic media. It amounts to: (1) showing that with suitable definitions of Φ and D, $\langle \mathcal{E} \rangle$ remains $\exp[-\frac{1}{2}\Phi^2]$ and Eq. (1.14) continues to hold, (2) finding a suitable definition for Ω and then showing that the boundaries of the saturated regimes are still given by Table 1, (3) showing that in the fully saturated regime the statistics of $\mathcal{E}$ are Gaussian and that in the partially saturated regime they are (for $p < 2$) as given in Table 3, (4) giving new formulas for the corrections to Gaussian statistics and coherence tails and (5) giving a method for computing $\Lambda(\omega)$. These steps will be carried out in order. In doing so it will be assumed that a ray approximation is valid for the unperturbed problem with $\mu = \mu_0$.

A. The First and Second Moments

The path integral for $\langle \mathcal{E} \rangle$ will contain a factor

$$\exp\left[-\frac{k^2}{2}\int_0^R dz \int_0^R dz' \, \rho\left(\vec{r}(z) - \vec{r}(z') + \vec{e}_z(z-z'), 0; \tfrac{1}{2}(\vec{r}(z) + \vec{r}(z')) + \tfrac{1}{2}\vec{e}_z(z+z')\right)\right] \tag{9.2}$$

The path dependence of the third argument of ρ will be approximated by setting $\frac{1}{2}(\vec{r}(z) + \vec{r}(z')) = \vec{s}(\bar{z})$ where $\bar{z} = \frac{1}{2}(z+z')$ and $\vec{s}$ is the unperturbed ray satisfying

$$\vec{s}''(z) + \vec{\nabla}\mu_0(\vec{s}(z) + \vec{e}_z z) = 0 \tag{9.3}$$

Here $\vec{s} = (s_x, s_y)$ is a two dimensional vector and $(\vec{s}(0), 0)$ and $(\vec{s}(R), R)$ are the source and receiver co-ordinates. If there is more than one unperturbed ray connecting the source to the receiver it is assumed that they are far enough apart that the path integral reduces to a sum of (statistically) independent terms coming from paths near each ray.[21] Defining a new path $\vec{u}(z)$ by $\vec{r}(z) = \vec{s}(z) + \vec{u}(z)$, the Markov approximation now amounts to setting

$$\vec{r}(z) - \vec{r}(z') + \vec{e}_z(z - z') \approx (\vec{s}'(\overline{z}) + \vec{e}_z)(z - z') \tag{9.4}$$

The essence of the approximation is neglecting $\vec{u}(z) - \vec{u}(z')$. By requirement (ii) the substitution $\vec{s}(z) - \vec{s}(z') \approx \vec{s}'(\overline{z})(z - z')$ is always valid. The reader will note that by (iii) $\vec{s}'$ is actually small compared to $\vec{e}_z$. However, in a sufficiently anisotropic medium $\vec{s}'$ cannot be neglected on the right-hand side of Eq. (9.4). Assuming for the moment the validity of the Markov approximation, the analog of $\hat{\rho}(0, 0)$ will be $\hat{\rho}(\vec{0}, 0; z)$ where

$$\hat{\rho}(\vec{0}, 0; z) = \int_{-\infty}^{\infty} \rho((\vec{s}'(z) + \vec{e}_z)\, u, 0; \vec{s}(z) + \vec{e}_z z)\, du \tag{9.5}$$

and the path integral for $\langle \mathcal{E} \rangle$, which is now trivial since ρ no longer contains the path $\vec{u}$, will yield $\langle \mathcal{E} \rangle = \mathcal{E}_0 \exp[-\frac{1}{2}\Phi^2]$ where

$$\Phi^2 = k^2 \int_0^R \hat{\rho}(\vec{0}, 0; z)\, dz \tag{9.6}$$

Continuing to assume the validity of the Markov approximation the next thing to compute is $<\mathcal{E}^*(2)\ \mathcal{E}(1)>$. There are two paths $\vec{r}_1 = \vec{s} + \vec{w}_1$ and $\vec{r}_2 = \vec{s} + \vec{w}_2$ where $\vec{s}$ satisfies Eq. (9. 3) with the boundary conditions $\vec{s}(0) = \frac{1}{2}(\vec{r}_{01} + \vec{r}_{02})$ and $\vec{s}(R) = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ and the approximation is

$$\int_0^R dz \int_0^R dz' \rho(\vec{r}_i(z) - \vec{r}_j(z') + \vec{e}_z(z - z'),\ t_i - t_j\ ;\ \tfrac{1}{2}(\vec{r}_i(z) + \vec{r}_j(z')) + \tfrac{1}{2}\vec{e}_z(z + z'))$$

$$\approx \int_0^R \hat{\rho}(\vec{w}_i(z) - \vec{w}_j(z)\ ;\ t_i - t_j\ ;\ z)\, dz \tag{9. 7}$$

for i, j = 1, 2 where

$$\hat{\rho}(\vec{w},\ t;\ z) = \int_{-\infty}^{\infty} \rho(\vec{w} + (\vec{s}\,'(z) + \vec{e}_z)u,\ t;\ \vec{s}(z) + \vec{e}_z z)\, du \tag{9. 8}$$

The path integral for $<\mathcal{E}^*(2)\ \mathcal{E}(1)>$ is then

$$<\mathcal{E}^*(2)\ \mathcal{E}(1)> = \frac{1}{4k^2} \int d^2(\text{paths}) \exp\left[i\, S_0(\text{path } 1) - i\, S_0(\text{path } 2) - \int_0^R d(\vec{w}_1(z) - \vec{w}_2(z),\ t_1 - t_2;\ z)\, dz\right] \tag{9. 9}$$

where

$$S_0 = k \int_0^R \left[\tfrac{1}{2}(\vec{r}\,'(z))^2 - \mu_0(\vec{r}(z) + \vec{e}_z z)\right] dz \tag{9. 10}$$

and

$$d(\vec{w}, t; z) = k^2\, [\,\hat{\rho}(\vec{0},\ 0; z) - \hat{\rho}(\vec{w},\ t;\ z)\,] \tag{9. 11}$$

Introducing paths $\vec{u} = \frac{1}{2}(\vec{w}_1 + \vec{w}_2)$ and $\vec{v} = \vec{w}_1 - \vec{w}_2$ we can, according to

(ii), expand S_0(path 1) - S_0(path 2) in powers of $\vec{u}$ and $\vec{v}$ and keep only the leading terms which are quadratic. Proceeding in this way yields[22]

$$\frac{\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle}{\mathcal{E}_0^*(2)\,\mathcal{E}_0(1)} = |2k\mathcal{E}_0|^{-2}\int d^2(\text{paths})\exp\left[ik\int_0^R(\vec{u}\,'(z)\cdot\vec{v}\,'(z) - u_i(z)\,v_j(z)\,\mu_{ij}(z))\,dz - \int_0^R d(\vec{v}(z),\, t_1 - t_2;\, z)\,dz\right] \tag{9.12}$$

where the two-by-two matrix (in $\vec{e}_x - \vec{e}_y$ space) $\mu_{ij}(z)$ is

$$\mu_{ij}(z) = \frac{\partial^2}{\partial x_i\,\partial x_j}\,\mu_0(\vec{x})\Big|_{\vec{x} = \vec{s}(z) + \vec{e}_z z} \tag{9.13}$$

The path $\vec{u}$ now appears only as a linear factor in the exponential and integrating over it will produce a product of δ-functions which force $\vec{v}(z)$ to be equal to the special path $\vec{\upsilon}(z)$ which satisfies the differential equation and boundary conditions[22]

$$\begin{aligned} \upsilon_i''(z) + \mu_{ij}(z)\,\upsilon_j(z) &= 0 \\ \vec{\upsilon}(0) &= \vec{r}_{01} - \vec{r}_{02} \\ \vec{\upsilon}(R) &= \vec{r}_1 - \vec{r}_2 \end{aligned} \tag{9.14}$$

Then setting $\vec{v}$ equal to $\vec{\upsilon}$ in d the remaining path integral just produces[17] $|2k\mathcal{E}_0|^2$ and one finds Eq. (1.14)

$$\langle \mathcal{E}^*(2)\,\mathcal{E}(1)\rangle = \mathcal{E}_0^*(2)\,\mathcal{E}_0(1)\,\exp[-\tfrac{1}{2}D] \tag{1.14''}$$

with

$$D = 2\int_0^R d(\vec{\upsilon}(z),\, t_1 - t_2;\, z)\,dz \tag{9.15}$$

The object D defined in Eq. (9.15) is just the phase structure function of first order geometric optics[1,7] for a general inhomogeneous anisotropic medium which satisfies (i), (ii) and (iii). Note that $\vec{\nu}$ is always linear in $\vec{r}_{01} - \vec{r}_{02}$ and $\vec{r}_1 - \vec{r}_2$. When μ_0 is a constant, $\vec{\nu}(z) = (\vec{r}_{01} - \vec{r}_{02})(R - z)/R + (\vec{r}_1 - \vec{r}_2)z/R$ and for a homogeneous isotropic medium Eq. (9.15) reduces to Eq. (1.16).

For an isotropic medium where ρ depends only on the magnitude of $\vec{x} - \vec{x}'$ the Markov approximation is valid whenever the parabolic wave equation is. The reason is the same as in Sec. (2). In App. (E) the formula for the first correction to the Markov approximation to $\langle \mathcal{E}^* \mathcal{E} \rangle$ is given. One can explicitly verify that the error is small when the parabolic wave equation is valid.

The situation for anisotropic media is more complicated. Consider an anisotropic but homogeneous medium with constant μ_0. Typical inhomogeneities will not be spherically symmetric and one needs to consider the three cases shown in Figs. (6a), (6b) and (6c). The asymmetric inhomogeneities introduce a new small angle θ_0, the ratio of the small dimension to the large one. Examining the error in the Markov approximation as given in App. (E) one finds that, for the case shown in Fig. (6a), the Markov approximation fails when the r.m.s. multiple scattering angle is of order θ_0. For the case shown in Fig. (6b), it fails when the r.m.s. multiple scattering angle is of order of the angle of incidence θ_i and for the situation in Fig. (6c), it fails when the r.m.s. multiple

scattering angle is of order unity, i. e., when the parabolic wave equation fails. Since θ_0 can be small compared to unity the Markov approximation can fail in an anisotropic medium before the parabolic wave equation does but only for some propagation paths. When it fails Eq. (1.14) is not valid and this represents a defect in the theory which is not easy to remove.

It should not be surprising that the Markov approximation can fail sooner in an anisotropic medium. The Markov approximation can be interpreted as the statement that the system has "no memory" in range, i. e., that scatterings at a given range point are independent of previous distant scatterings. In an isotropic medium this will be true as long as the r. m. s. multiple scattering angle is small and the wave keeps moving in the same direction. However, in an anisotropic medium, when the scattering by a given inhomogeneity can be highly dependent on the angle of incidence, a distant scattering which has deflected the wave only through a small angle will not be "forgotten." For the inhomogeneities shown in Fig. (6) the scattering is strongly dependent on angle of incidence (measured from the long axis of the inhomogeneities) when the angle is of order θ_0. When the incident wave is along the long axis as in Fig. (6a), it begins to remember previous scatterings when the scattering angle builds up to θ_0 and the pieces of the wave have incidence angles greater than θ_0. For the case shown in Fig. (6b) the past history of the wave becomes important when pieces of the wave have been deflected by θ_i and are incident along the long axis. When θ_i

approaches $\pi/2$ as in Fig. (6c) the process has no memory as long as the r.m.s. multiple scattering angle is less than unity.

Yet another way to understand the peculiarities of anisotropic media is to return to the remarks following Eqs. (6.8) and (6.9). For an isotropic medium the average of $\dot{\mu}$ integrated along a Fermat path $\vec{w}$ will be the same as the average of $\dot{\mu}$ integrated along the unperturbed ray $\vec{s}$ as long as the r.m.s. multiple scattering angle is small. However, in an anisotropic medium the average of $\dot{\mu}$ integrated along a path can be very sensitive to the local direction $\vec{w}'$ of the path. In fact, for the situation shown in Fig. (6a), the average of $\dot{\mu}$ integrated along a Fermat path deviates from the average along an unperturbed ray as soon as $|\vec{w}'| \sim \theta_0$ and for the situation in Fig. (6b) when $|\vec{w}'| \sim \theta_i$. This leads to the same criteria as before.

The combination of an anisotropic medium and a spacially varying $\mu_0(\hat{x})$ leads to a new set of complications. This will be illustrated for propagation in a channel where the unperturbed rays make loops as shown in Fig. (7) and where the long axis of the inhomogeneities is parallel to the channel axis. The medium will also be assumed to be statistically homogeneous in the direction of the channel axis but not necessarily in the transverse directions. (This is a prototype of the physical situation which occurs for sound propagation in the ocean.[7]) The scattering will be strongest when the tangent to the unperturbed ray is pointing along the long axis of the inhomogeneities, i.e., at the turning points. For small θ_0 one can in fact ignore all of the propagation path except for a set of discrete regions around turning points where the tangent to the ray is within θ_0 of the channel axis. Assuming that a Markov

approximation is valid for propagation through one of these regions, it will also be valid for propagation through many turning points provided only that the average scattering at a given turning point is at most weakly dependent on scatterings at previous turning points. Assuming that the turning points are separated by more than a coherence length the effect of previous scatterings will be a random modulation of the range z_0 and (transverse) location $\vec{s}_0$ of a turning point. Now the average scattering around a turning point is dependent only on its location $\vec{s}_0$ in the channel and not on its range z_0. Thus the Markov approximation will be valid out to range such that random variations in $\vec{s}_0$ are big enough to change the average scattering. This turns out to be a much longer range[7] than that for which the r.m.s. multiple scattering angle (which is dominated by variations in z_0) becomes of order θ_0. The extended validity of the Markov approximation can be demonstrated explicitly using the Fermat path formalism of Sec. (6). One works out the properties of Fermat paths which are randomly deflected at turning points and then compares averages of μ integrated along these paths to averages of μ integrated along the unperturbed ray. For a given channel one can then find out when the Markov approximation will break down. The result is just the criteria stated above.

B. The Saturated Regimes

It will temporarily be assumed that the medium has a single

scale. Then in an anisotropic inhomogeneous medium the scale length L becomes a z-dependent two-by-two matrix (in $\vec{e}_x - \vec{e}_y$ space) defined by the expansion of $\hat{\rho}$ [22]

$$\hat{\rho}(\vec{w}, 0; z) = \hat{\rho}(\vec{0}, 0; z)[1 - \tfrac{1}{2}(L^{-2}(z))_{ij}\, w_i w_j + 0(|\vec{w}|^3)] \tag{9.16}$$

The first task in discussing the saturated regimes is to find the correct definition of Ω and establish their boundaries. The general definition of Ω will involve L and some geometric parameters associated with the unperturbed problem. From Secs. (3) and (7) one can see that Ω measures the rate at which the phase of the oscillating factor in the path integral varies as a path moves away from an unperturbed ray. To examine this in more detail consider paths that leave the source at $z = 0$, go to the receiver at $z = R$ and at some point z_0 in between are separated from the unperturbed ray $\vec{s}(z_0)$ by $\vec{\ell}$. Let $\overline{S}(\vec{\ell}, z_0)$ be the minimum of $S_0(\text{path}) - S_0(\text{unperturbed ray})$ taken over all paths of this class. The minimum is achieved for a path that follows an unperturbed ray from the source to $(\vec{s}(z_0) + \vec{\ell}, z_0)$ and then another unperturbed ray from $(\vec{s}(z_0) + \vec{\ell}, z_0)$ to the receiver. When μ_0 is a constant $\overline{S}(\vec{\ell}, z_0)$ is simply

$$\overline{S}(\vec{\ell}, z_0) = \tfrac{1}{2}(\vec{\ell})^2 B(z_0) \tag{9.17}$$

where B which already appeared in Eq. (6.11) is

$$B(z_0) = \frac{kR}{z_0(R - z_0)} \qquad (9.$$

and Ω^{-1} is the average of $B^{-1}L^{-2}$, i.e.

$$\Omega^{-1} = L^{-2}\,\overline{B^{-1}} = L^{-2}\left[\frac{1}{R}\int_0^R B^{-1}(z_0)\,dz_0\right] \tag{9.19}$$

Thus Ω is a measure of the phase change required to move a path a distance L away from the unperturbed ray. In general there is a two-by-two matrix B defined by the expansion for small $\vec{\ell}$ [22]

$$\overline{S}(\vec{\ell}, z_0) = \frac{1}{2}\,\ell_i\,\ell_j\,B_{ij}(z_0) + 0(|\vec{\ell}|^3) \tag{9.20}$$

and Ω^{-1} will be an average of $L^{-2}B^{-1}$. It is convenient to weight the average by $\hat{\rho}(0, 0; z)$ and Ω will be defined as [22]

$$\Omega^{-1} = \frac{1}{2}\,\overline{(L^{-2})_{ij}(B^{-1})_{ij}} = \frac{1}{2}\,\frac{\int_0^R \hat{\rho}(\vec{0}, 0; z)\,(L^{-2}(z))_{ij}\,(B^{-1}(z))_{ij}\,dz}{\int_0^R \hat{\rho}(\vec{0}, 0; z)\,dz} \tag{9.21}$$

With this definition of Ω one can follow through the arguments of Sec. (3) and verify that saturation and Gaussian statistics are expected when $\Phi > 1$ and $\Phi/\Omega > 1$. A more precise procedure is to compute $(\langle I^2\rangle - \langle I\rangle^2)/\langle I\rangle^2$ in the Rytov approximation to find the boundary of the saturated regime and then in the saturated regime compute the corrections to Gaussian statistics and verify that they are small. A straightforward evaluation of $(\langle I^2\rangle - \langle I\rangle^2)/\langle I\rangle^2\Big|_{\text{Rytov}}$ shows that it does in fact exceed unity when $\Phi > 1$ and $\Phi/\Omega > 1$ indicating that the boundary is correct. Using the formulas of App. F one can verify that in the saturated regime the corrections to Gaussian statistics are indeed small.

With the appropriate change in the definition of $(L^{-2})_{ij}$ for $p < 2$, the same procedure can be extended to media with multiple scales. The result is that with Ω defined as in Eq. (9. 21) the boundaries given in Table 1 remain correct and that for $p < 2$ the partially saturated statistics given in Table 3 also remain correct.

To actually calculate $B_{ij}(z)$ the following result is useful. Define a Green's function $g_{ij}(z, z')$ by [22]

$$\frac{\partial^2}{\partial z^2} g_{ij}(z, z') + \mu_{ik}(z)\, g_{kj}(z, z') = \delta_{ij}\, \delta(z - z')$$

$$g_{ij}(0, z') = g_{ij}(R, z') = 0 \qquad (9.22)$$

Then it is straightforward to verify that

$$(B^{-1}(z))_{ij} = -g_{ij}(z, z) \qquad (9.23)$$

C. <u>Correlations in Frequency</u>

In general one can write

$$\frac{<\mathcal{E}^*(\omega')\mathcal{E}(\omega)>}{\mathcal{E}_0^*(\omega')\ \mathcal{E}_0(\omega)} = \exp\left[-\frac{1}{2}\left(\frac{\omega - \omega'}{\omega_g}\right)^2\right] \Lambda(\omega - \omega') \qquad (1.19')$$

where the exponential factor comes from geometric optics and Λ is to be computed from the path integral. The geometric optics decorrelation frequency ω_g is

$$\omega_g^{-2} = \left\langle \left(\frac{d}{d\omega} \int_0^R k \left[\mu_\omega(\vec{s}_\omega(z) + \vec{e}_z z, t) - <\mu_\omega(\vec{s}_\omega(z) + \vec{e}_z z, t)> \right] dz \right)^2 \right\rangle \qquad (9.24)$$

and for a general dispersive medium both μ and the unperturbed ray $\vec{s}$

will depend on ω. For a nondispersive medium ω_g is equal to $\overline{\omega}/\Phi$ where Φ is evaluated at the central frequency $\overline{\omega}$.

As before the path integral for Λ is tractable only for media with $p > 2$. The derivation proceeds as in Sec. (3) and after introducing scaled paths $\vec{\xi} = \left(\frac{\overline{k}^2 c_g}{2(\omega - \omega')}\right)^{\frac{1}{2}} (\vec{v} - \vec{s}_{\overline{\omega}})$ where $\overline{\omega} = \frac{1}{2}(\omega + \omega')$ and c_g is the group velocity at $\omega = \overline{\omega}$ the path integral for Λ gives

$$\Lambda(\omega) = \frac{K(\omega)}{K(0)} \tag{9.25}$$

where [22]

$$K(\omega) = \int d(\text{paths}) \exp\left[-i \int_0^R [(\vec{\xi}'(z))^2 - \xi_i(z)\, \xi_j(z) (\mu_{ij}(z) + i\omega h_{ij}(z))]\, dz\right] \tag{9.26}$$

with

$$h_{ij}(z) = c_g^{-1}\, \hat{\rho}(\vec{0}, 0; z) (L^{-2}(z))_{ij} \tag{9.27}$$

If the path integral for K is written out in its finite form it becomes an ordinary integral of large dimension whose integrand is the exponential of a quadratic form. Such an integral is proportional to one over the square root of the determinant of the quadratic form and in particular Λ will be the square root of the ratio of two determinants. As the number of integration points goes to infinity the determinants become functional determinants. There are two equivalent methods[6] for computing the ratio of these functional determinants.

In the first method one has to find all the eigenvalues ω_n of the differential equation

$$\xi_i^{(n)''}(z) + \mu_{ij}(z)\,\xi_j^{(n)}(z) - \omega_n h_{ij}(z)\,\xi_j^{(n)}(z) = 0 \qquad (9.28)$$

subject to the boundary conditions $\vec{\xi}^{(n)}(0) = \vec{\xi}^{(n)}(R) = 0$. Having done this $\Lambda(\omega)$ is

$$\Lambda(\omega) = \left(\prod_n \frac{1}{1 + i\frac{\omega}{\omega_n}} \right)^{\frac{1}{2}} \qquad (9.29)$$

In the second method one defines a two-by-two matrix $M_{ij}(z, \omega)$ by the differential equation

$$M''_{ij}(z, \omega) + \mu_{ik}(z)\, M_{kj}(z, \omega) + i\,\omega\, h_{ik}(z)\, M_{kj}(z, \omega) = 0 \qquad (9.30)$$

and boundary conditions[22] in z

$$\begin{aligned} M_{ij}(0, \omega) &= 0 \\ M'_{ij}(0, \omega) &= \delta_{ij} \end{aligned} \qquad (9.31)$$

Then Λ is given by the ratio of determinants

$$\Lambda(\omega) = \left(\frac{\det M(R, 0)}{\det M(R, \omega)} \right)^{\frac{1}{2}} \qquad (9.32)$$

As an example of how Λ is computed consider a homogeneous isotropic medium where $\mu_{ij} = 0$ and $h_{ij} = \delta_{ij}\, c_g^{-1}\, \hat{\rho}(0, 0)\, L^{-2}$. The eigenfunctions of the operator in Eq. (9.28) are then of the form $\delta_{i1} \sin(n_1 \pi z/R)$ and $\delta_{j2} \sin(n_2 \pi z/R)$ and the eigenvalues are $-n_1^2 \pi^2 \omega_1$ and $-n_2^2 \pi^2 \omega_1$ where $\omega_1 = c_g L^2/R^2\, \hat{\rho}(0, 0)$ as in Sec. (4). The infinite product in Eq. (9.29) is then a product over two sets of integers

$$\Lambda(\omega) = \left(\prod_{n_1=1}^{\infty} \frac{1}{1 - i\frac{\omega}{n_1^2 \pi^2 \omega_1}} \prod_{n_2=1}^{\infty} \frac{1}{1 - i\frac{\omega}{n_2^2 \pi^2 \omega_1}} \right)^{\frac{1}{2}} \tag{9.33}$$

and the two equal factors just cancel the square root. The result is

$$\Lambda(\omega) = \prod_{n=1}^{\infty} \frac{1}{1 - i\frac{\omega}{n^2 \pi^2 \omega_1}} = \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{1 + \left(\frac{i\omega}{\omega_1}\right)^{\frac{1}{2}} \frac{1}{n\pi}} = \frac{\left(\frac{i\omega}{\omega_1}\right)^{\frac{1}{2}}}{\sin\left(\frac{i\omega}{\omega_1}\right)^{\frac{1}{2}}} \tag{9.34}$$

and with $\omega_1 = \omega_0 \alpha/6$ Eq. (1.21) is reproduced. To compute Λ by the second method one finds immediately that $M_{ij}(z, \omega) = \delta_{ij} R\left(\frac{\omega_1}{i\omega}\right)^{\frac{1}{2}} \sin\left(\frac{z}{R}\left(\frac{i\omega}{\omega_1}\right)^{\frac{1}{2}}\right)$, $M_{ij}(z, 0) = \delta_{ij} z$ and Eq. (9.32) yields the expected answer.

Once ω_g and Λ have been determined everything proceeds as in the homogeneous isotropic case. In particular $\mathcal{C}$ satisfies Gaussian statistics in the fully saturated regime and in the partially saturated regime for $p < 2$ (where Λ is unfortunately not known) it is a phase times a Gaussian. There is one new point worth mentioning. In the calculation of Sec. (4) $< \mathcal{l}(\tau) >$ vanished for $\tau < 0$ because Λ was analytic in the upper half plane. When $\mu_{ij}(z)$ is non-zero there can be a finite number of positive eigenvalues ω_n. Then Λ is no longer analytic in the upper half plane and $< \mathcal{l}(\tau) >$ is nonvanishing for $\tau < 0$. This in fact happens for propagation of sound in the ocean.[7]

The transition to inhomogeneous anisotropic media has now been completed. The reader who is interested in seeing how the method works in detail for a realistic problem can consult the book of Flatte, et al.[7]

10. Conclusions

The path integral has turned out to be a powerful tool. It has provided a precise, (very nearly) complete and global picture of what goes on in the saturated regimes. The unsaturated regime where the Rytov approximation is valid could also be treated by path integral methods. While this would lead to a more unified picture, in the end it would only amount to a rederivation of the Rytov approximation. A more fruitful endeavor would be to make an attack on the remaining unsolved problems in the saturated regimes. For situations where a scalar wave equation is sufficient and the (multiple) scattering angles are small the remaining problems are:

a) How to compute (except numerically) the coherence in frequency, $\Lambda(\omega)$, for multiple scale media with $p < 2$.

b) What are the detailed (beyond those given in App.(D)) statistics of $\mathcal{E}$ in the partially saturated regime for $4 > p > 2$?

c) How to compute the second moment $\langle \mathcal{E}^*(2)\, \mathcal{E}(1) \rangle$ for those propagation paths in highly anisotropic media where the Markov approximation is not valid?

d) What is the detailed behavior of $\mathcal{E}$ at the boundaries between the unsaturated and saturated regimes and between the fully and partially saturated regimes?

These are difficult problems which may not have any simple solution and, in particular, the path integral may not be the best method for attacking them. On the other hand, it is quite remarkable that the use of Feynmann's path integral has reduced the problem to a few unknowns which occur only in special cases.

Among the other methods for treating wave propagation in random media, the most powerful ones use the Markov approximation from the beginning. With the Markov approximation one can derive local partial differential equations for the moments of $\mathcal{E}$.[3-5] These equations have been studied extensively, especially by the Russian school.[3,4] In the Markov approximation the path integrals for the moments are formal solutions to these partial differential equations. The equations for the first and second moments can be integrated analytically and correspondingly the path integrals can be done analytically. For the higher moments, the differential equations have yielded only some information about[4] $\langle I(1)\ I(2) \rangle$. The reason that this approach has not yielded more is that to determine the asymptotic (long-range) behavior of a function from its defining partial differential equation is highly non-trivial. The path integral has the advantage that it works on a global rather than local level, making it easier to determine the asymptotics.

The reader who is familiar with Mercier's[12] treatment of the phase screen problem (an idealized case where all the scattering takes place on a thin sheet) will have noticed the similarity between his methods and those used here. The similarity is partly just the mathematics of manipulating integrals but there is also a physical reason. Any medium can be approximated by a (perhaps) large but finite number of phase screens. The wave field can then be expressed as a large but finite dimensional integral over the surfaces of the screens. But this is just the path integral in its finite form. Thus the

path integral can be thought of as a scheme where one approximates the medium by n phase screens and then letting n go to infinity recovers the original problem.

APPENDIX A CORRECTIONS TO THE MARKOV APPROXIMATION

The exact path integral for $\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle$ can be expanded as

$$\langle \mathcal{E}^*(2)\mathcal{E}(1)\rangle = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{1}{4k^2}\int d^2(\text{paths})\exp\left[\frac{ik}{2}\int_0^R\left[\left(\vec{r}_1{}'(z)\right)^2 - \left(\vec{r}_2{}'(z)\right)^2\right]dz - V_o\right]$$

$$\times [V_o - V]^m \qquad \text{(A.1)}$$

where V is given by Eq. (2.10) and V_o is the Markov approximation given by Eq. (2.11). The $m = 0$ term is the Markov approximation and the $m = 1$ term will be computed below for the special case $\langle \mathcal{E}^*(1)\mathcal{E}(1)\rangle$.

For $\langle \mathcal{E}^*(1)\mathcal{E}(1)\rangle = \langle I\rangle$ the first correction contains $[V_o - V]$ which can be replaced by $-V$ because, as may be seen from Sec. (2) the piece proportional to V_o vanishes. We then have to first order

$$\langle I\rangle = \langle I\rangle_0 - \frac{1}{4k^2}\int d^2(\text{paths})\exp\left[ik\int_0^R \vec{v}'(z)\cdot\vec{u}'(z)dz - V_o\right]V \qquad \text{(A.2)}$$

where paths $\vec{v}(z) = \vec{r}_1(z) - \vec{r}_2(z)$ and $\vec{u}(z) = \frac{1}{2}\left(\vec{r}_1(z) + \vec{r}_2(z)\right)$ have been introduced. In terms of the Fourier transform $\tilde{\rho}$ of ρ (Eq. (5.3)) V is

$$V = 2k^2\int_0^R dz_1\int_0^R dz_2\int d^2q\,dq_z\,\tilde{\rho}\left(\sqrt{\vec{q}^{\,2} + q_z^2},0\right)e^{iq_z(z_1 - z_2)}$$

$$\times e^{i\vec{q}\cdot\left(\vec{u}(z_1) - \vec{u}(z_2)\right)}\sin\left[\tfrac{1}{2}\vec{q}\cdot\vec{v}(z_1)\right]\sin\left[\tfrac{1}{2}\vec{q}\cdot\vec{v}(z_2)\right] \qquad \text{(A.3)}$$

where $\vec{q} = (q_x, q_y)$ is a two-dimensional vector. Writing

$$e^{i\vec{q}\cdot\left(\vec{u}(z_1) - \vec{u}(z_2)\right)} = \exp\left[i\vec{q}\cdot\int_0^R \vec{u}(z)\,[\delta(z - z_1) - \delta(z - z_2)]dz\right]$$

and inserting V as given by Equation (A.3) into Eq. (A.2) one finds that since V_o depends only on $\vec{v}$ the integral over $\vec{u}(z)$ can be done and that it leads to a δ- functional which forces $\vec{v}$ to satisfy the equation

$$\vec{v}''(z) = \frac{\vec{q}}{k}[\delta(z - z_1) - \delta(z - z_2)] \tag{A.4}$$

with the boundary conditions $\vec{v}(0) = \vec{v}(R) = 0$. In terms of the Greens function $g(z,z')$ which satisfies $\frac{\partial^2}{\partial z^2} g(z,z') = \delta(z - z')$ and $g(0,z') = g(R,z') = 0$, $\vec{v}(z)$ is constrained to be $\vec{v}_o(z)$ where

$$\vec{v}_o(z) = \frac{\vec{q}}{k}\Big(g(z,z_1) - g(z,z_2)\Big) \tag{A.5}$$

The path integral is then done by replacing $\vec{v}(z)$ by $\vec{v}_o(z)$ in V_o and in the representation of V and the final result is

$$-\frac{\langle I\rangle - \langle I\rangle_0}{\langle I\rangle_0} = 2k^2\int_0^R dz_1 \int_0^R dz_2 \int d^2q\,dq_z\, \tilde{\rho}\left(\sqrt{\vec{q}^{\,2} + q_z^2}, 0\right) e^{iq_z(z_1 - z_2)}$$

$$\times \sin\left[\tfrac{1}{2}\vec{q} \cdot \vec{v}_o(z_1)\right] \sin\left[\tfrac{1}{2}\vec{q} \cdot \vec{v}_o(z_2)\right] \exp\left[-\int_0^R d\left(|\vec{v}_o(z)|, 0\right) dz\right] \tag{A.6}$$

To estimate the size of the integral in Eq. (A.6), one notes that $|z_1 - z_2|$ will be restricted to order L or less and that for $|z_1 - z_2| \sim L$, $\vec{q} \cdot \vec{v}_0(z)$ is of order $q^2 L/k \sim q/k$ which is assumed to be small. The sines can then be expanded and using the identity

$$[g(z_1,z_1) - g(z_1,z_2)][g(z_2,z_1) - g(z_2,z_2)] = -\frac{(z_1 - z_2)^2 g(z_1,z_2)}{R} \tag{A.7}$$

one finds

$$\frac{\langle I\rangle - \langle I\rangle_0}{\langle I\rangle_0} = \frac{1}{2R}\int_0^R dz_1 \int_0^R dz_2 \int d^2q\,dq_z\; \tilde{\rho}\left(\sqrt{\vec{q}^{\,2} + q_z^2},0\right)\vec{q}^{\,4}(z_1 - z_2)^2$$

$$\times\, g(z_1,z_2)\exp\left[iq_z(z_1 - z_2) - \int_0^R d\left(|\vec{v}_o(z)|,0\right)dz\right] \quad \text{(A.8)}$$

Ignoring the term $\int d(|\vec{v}_0(z)|,0)dz$ in the exponential which can only make the integral smaller, changing to variables $u = z_1 - z_2$ and $\bar{z} = \frac{1}{2}(z_1 + z_2)$, approximating their limits by $-\infty < u < \infty$ and $0 < \bar{z} < R$ and setting $g(z_1, z_2) \approx g(\bar{z}, \bar{z})$ then yields

$$\frac{\langle I\rangle - \langle I\rangle_0}{\langle I\rangle_0} = -\frac{R\hat{\rho}(0,0)}{3L^2} \sim -\frac{R\langle\mu^2\rangle}{3L} \quad \text{(A.9)}$$

The correction to Markov approximation to $\langle\mathcal{E}^*(2)\,\mathcal{E}(1)\rangle$ can be analyzed in the same way. It is fractionally small as long as $\langle \mu^2\rangle R/L$ is small.

APPENDIX B CORRECTIONS TO GAUSSIAN STATISTICS

To begin with something simple, consider $\langle I^2 \rangle$. It is given by the path integral in Eq. (3.1) with $t_i = t_j$ and the end point conditions $\vec{r}_i(0) = \vec{r}_i(R) = 0$. Changing variables to $\vec{u}_1$, $\vec{u}_2$, $\vec{v}_1$ and $\vec{v}_2$ defined by

$$\vec{r}_1(z) = \vec{u}_1(z) + \tfrac{1}{2}\vec{v}_2(z) + \tfrac{1}{4}\vec{u}_2(z) + \tfrac{1}{2}\vec{v}_1(z)$$

$$\vec{r}_2(z) = \vec{u}_1(z) + \tfrac{1}{2}\vec{v}_2(z) - \tfrac{1}{4}\vec{u}_2(z) - \tfrac{1}{2}\vec{v}_1(z)$$

$$\vec{r}_3(z) = \vec{u}_1(z) - \tfrac{1}{2}\vec{v}_2(z) + \tfrac{1}{4}\vec{u}_2(z) - \tfrac{1}{2}\vec{v}_1(z)$$

$$\vec{r}_4(z) = \vec{u}_1(z) - \tfrac{1}{2}\vec{v}_2(z) - \tfrac{1}{4}\vec{u}_2(z) + \tfrac{1}{2}\vec{v}_1(z) \qquad \text{(B.1)}$$

the integral over $\vec{u}_1(z)$ can be done and it produces a δ-functional which forces $\vec{u}_2''(z)$ to vanish everywhere. With the end point conditions $\vec{u}_2(0) = \vec{u}_2(R) = 0$, the only solution is $\vec{u}_2(z) = 0$. The quadruple path integral then reduces to the double path integral over paths $\vec{v}_1$ and $\vec{v}_2$

$$\frac{\langle I^2 \rangle}{\langle I \rangle} = \frac{1}{4k^2}\int d^2(\text{paths})\exp\left[ik\int_0^R \vec{v}_1{}'(z) \cdot \vec{v}_2{}'(z)dz - M\right] \qquad \text{(B.2)}$$

where M was defined in Eq. (3.2) and for $\vec{u}_2 = 0$ it is explicitly

$$M = \int_0^R \Big[2d\left(|\vec{v}_1(z)|,0\right) + 2d\left(|\vec{v}_2(z)|,0\right) - d\left(|\vec{v}_1(z) + \vec{v}_2(z)|,0\right)$$
$$- d\left(|\vec{v}_1(z) - \vec{v}_2(z)|,0\right)\Big] dz \qquad \text{(B.3)}$$

The two regions (a) and (b) discussed in Section III are $|\vec{v}_1(z)| < L/\Phi$ with $\vec{v}_2$ arbitrary and $|\vec{v}_2(z)| < L/\Phi$ with $\vec{v}_1$ arbitrary.

It was pointed out in the text that M is of order unity or smaller throughout regions (a) and (b). Actually, there is a further region

(having some overlap with (a) and (b)) where M can be small. It is (c) $|\vec{v}_1(z)| < L/\sqrt{\Phi}$, $|\vec{v}_2(z)| < L/\sqrt{\Phi}$ and owes its existence to the fact that when both $|\vec{v}_1(z)|$ and $|\vec{v}_2(z)|$ are small, M is quartic in the $\vec{v}$'s. In all other regions of path space, e^{-M} is exponentially small.

Our first task is to dispose of the extra region (c) by showing that for small α the volume of path space occupied by this region is exponentially small compared to the volumes occupied by regions (a) and (b). An estimate of the volume of path space occupied by region (a) is

$$\frac{1}{4k^2}\int d^2(\text{paths})\ \exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)dz - \frac{\Phi^2}{L^2R}\int_0^R \left(\vec{v}_1(z)\right)^2 dz\right] = 1 \qquad \text{(B.4)}$$

where the integral is done by integrating over $\vec{v}_2(z)$ which produces a δ-functional that forces $\vec{v}_1(z)$ to vanish. An estimate of the volume occupied by region (c) is

$$\frac{1}{4k^2}\int d^2(\text{paths})\ \exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)dz - \frac{\Phi}{2RL^2}\int_0^R\left(\vec{v}_1^{\,2}(z) + \vec{v}_2^{\,2}(z)\right) dz\right]$$

$$= \frac{(6/\alpha)}{\sinh^2\sqrt{\frac{3}{\alpha}} + \sin^2\sqrt{\frac{3}{\alpha}}} \sim \frac{24}{\alpha}\, e^{-2\sqrt{\frac{3}{\alpha}}} \qquad \text{(B.5)}$$

where the value of the path integral is taken from Ref. (6). For small α, the volume occupied by region (c) is therefore exponentially small compared to the volume occupied by regions (a) and (b). This result, which may surprise some readers, deserves an explanation. In region (a) where $|\vec{v}_1|$ is always less than L/Φ, the factor $\exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)\,dz\right]$ $= \exp\left[-ik\int_0^R \vec{v}_1(z)\cdot\vec{v}_2''(z)dz\right]$ will restrict $|\vec{v}_2''|$ to values less than $\Phi/(kLR)$. Typical values of $|\vec{v}_2|$ will then be $R^2|\vec{v}_2''|/6 \sim L/\alpha$. At a

given range point z_o, the variables $\vec{v}_1(z_o)$ and $\vec{v}_2(z_o)$ span a four-dimensional space. In this space the volume occupied by paths in region (a) is roughly $L^4/(\Phi\alpha)^2$. At the same point the volume occupied by paths in region (c) is roughly L^4/Φ^2. Thus at each point z_o, the volume associated with region (c) is a factor of α^2 smaller than that occupied by region (a). To compute the total volume in path space, one has to multiply together the volumes at each range point z_o, taking into account the fact that the paths cannot bend too rapidly. The path integrals in Equations (B.4) and (B.5) do just this. The resulting exponential ratio of volumes should no longer be a surprise since at each range point the ratio is down by α^2.

It is therefore sufficient to consider only paths lying in regions (a) and (b). The fact that integrating separately over regions (a) and (b) leads to a slight over-counting can also be ignored. The volume in path space where regions (a) and (b) overlap is even smaller than the volume occupied by region (c). Now as was pointed out in the Section III, for most paths in region (a) $M \approx M_o^{(a)}$ where

$$M_o^{(a)} = 2\int_0^R d\left(|\vec{v}_1(z)|, 0\right) dz \tag{B.6}$$

and for most paths in region (b) $M \approx M_o^{(b)}$ with

$$M_o^{(b)} = 2\int_0^R d\left(|\vec{v}_2(z)|, 0\right) dz \tag{B.7}$$

The path integral in Eq. (B.2) can then be expanded according to

$$\int d^2(\text{paths})\exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)dz - M\right] \sim$$

$$\sum_{m=0}^{\infty}\int d^2(\text{paths})\exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)dz - M_o^{(a)}\right]\frac{\left(M_o^{(a)} - M\right)^m}{m!}$$

$$+\sum_{m=0}^{\infty}\int d^2(\text{paths})\exp\left[ik\int_0^R \vec{v}_1'(z)\cdot\vec{v}_2'(z)dz - M_o^{(b)}\right]\frac{\left(M_o^{(b)} - M\right)^m}{m!} \quad (B.8)$$

which is an asymtotic series in α. It is not a convergent series because (exponentially small) contributions from region (c) and the overlap of regions (a) and (b) are not being treated correctly. The $m = 0$ terms correspond to Rayleigh statistics and the $m = 1$ terms are the first correction. They will be computed explicitly below.

Eq. (B.8) generalizes to an arbitrary correlation in the obvious way. For a 2nth order moment there are n! important regions of path space. In each such region there is an M_o given by the analog of Eq. (3.4) or (3.5). The generalization of Eq. (B.8) is then a sum of n! terms, each of which is a series of powers of the appropriate $M_o - M$.

The path integrals for the $m = 1$ terms in Eq. (B.8) can be evaluated by inserting a spectral representation for $M_o - M$. If $\tilde{\rho}$ is the three-dimensional Fourier transfor of ρ (see Eq. (5.3)), then in region (a)

$$M_o^{(a)} - M = 4\pi k^2\int_0^R dz\int d^2q\ \tilde{\rho}(|\vec{q}|)e^{i\vec{q}\cdot\vec{v}_2(z)}\left[1 - \cos\left(\vec{q}\cdot\vec{v}_1(z)\right)\right] \quad (B.9)$$

and

$$\frac{1}{4k^2}\int d^2(\text{paths}) \exp\left[ik\int_0^R \vec{v}_1{}'(z)\cdot\vec{v}_2{}'(z)\,dz - M_o^{(a)}\right]\left[M_o^{(a)} - M\right] =$$

$$\pi\int_0^R dz'\int d^2q\;\tilde{\rho}(|\vec{q}|)\int d^2(\text{paths}) \exp\left[-ik\int_0^R \vec{v}_2(z)\cdot\left(\vec{v}_1{}''(z) - \frac{\vec{q}}{k}\,\delta(z - z')\right)dz - M_o^{(a)}\right]$$

$$\times\left[1 - \cos\left(\vec{q}\cdot\vec{v}_1(z')\right)\right] \tag{B.10}$$

Since M_o^a depends only on $\vec{v}_1$, the integral over $\vec{v}_2$ can be done and it produces a δ-functional which forces $\vec{v}_1{}''(z) - \frac{\vec{q}}{k}\,\delta(z - z')$ to vanish. In terms of the Greens function $g(z,z')$ defined by

$$\frac{\partial^2}{\partial z^2}\, g(z,z') = \delta(z - z') \tag{B.11}$$

with boundary conditions $g(0,z') = g(R,z') = 0$, $\vec{v}_2(z)$ is constrained to be $\frac{\vec{q}}{k}\,g(z,z')$. In $M_o^{(a)}$ and $\cos\left(\vec{q}\cdot\vec{v}_1(z')\right)$ one can then set $\vec{v}_1(z)$ equal to $\frac{\vec{q}}{k}\,g(z,z')$ and the remaining path just gives $\langle I\rangle$. The calculation of the correction in region (b) is identical and to leading order in α

$$\frac{\langle I^2\rangle - 2\langle I\rangle^2}{\langle I\rangle^2} = 4\pi k^2\int_0^R dz\int d^2q\;\tilde{\rho}(|\vec{q}|)\;Q(z,|\vec{q}|) \tag{B.12}$$

with

$$Q(z,|\vec{q}|) = 2\left[1 - \cos\left(\frac{\vec{q}^{\,2}}{k}\,g(z,z)\right)\right]\exp\left[-2\int_0^R d\left(\frac{|\vec{q}|}{k}\,g(z,z'),0\right)dz'\right] \tag{B.13}$$

An examination of the integral on the right-hand side of Eq. (B.12) shows that for small α, (i) $d\left(\frac{|\vec{q}|}{k} g(z,z'),0\right)$ can be approximated by $\frac{1}{2}k^2\hat{\rho}(0,0)\left(\frac{\vec{q}g(z,z')}{kL}\right)^2$, (ii) $2\left[1 - \cos\left(\frac{\vec{q}^{\,2}}{k} g(z,z)\right)\right]$ can be set equal to $\left(\frac{\vec{q}^{\,2}}{k} g(z,z)\right)^2$ and (iii) the dominant contribution comes from the regions $z \approx 0$ and $z \approx R$ where $g(z,z') \approx z(R - z')/R$ and $z'(R - z)/R$. The contributions from the regions $z \approx 0$ and $z \approx R$ are the same and

$$\frac{\langle I^2\rangle - 2\langle I\rangle^2}{\langle I\rangle^2} = 8\pi k^2 \int_0^\infty dz \int d^2q\, \tilde{\rho}(|\vec{q}|)\left(\frac{\vec{q}^{\,2} z}{k}\right)^2 \exp\left[-\frac{R\hat{\rho}(0,0)z^2\vec{q}^{\,2}}{3L^2}\right]$$

$$= \alpha\left(\frac{3\pi}{4}\right)^{\frac{1}{2}} \frac{L\int_0^\infty q^2\tilde{\rho}(q,0)\,dq}{\int_0^\infty q\tilde{\rho}(q,0)\,dq} \tag{B.14}$$

The correction to $\langle I^n\rangle$ involves n! regions of path space and in each of these regions there are $n(n - 1)/2$ terms in $M_o - M$ which differ only by permutations of the paths. The result is that the correction to $\langle I^n\rangle$ is $n!(n)(n - 1)/4$ times the correction to $\langle I^2\rangle$.

Moving on to a more complicated object, consider $\langle I(1)\,I(2)\rangle$. It is given by the path integral in Eq. (B.2) but the end point conditions on the paths are now $\vec{v}_1(0) = \vec{v}_1(R) = 0$ and $\vec{v}_2(0) = \vec{r}_{o1} - \vec{r}_{o2}$, $\vec{v}_2(R) = \vec{r}_1 - \vec{r}_2$ and now M is

$$M = \int_0^R \Big[2d\left(|\vec{v}_1(z)|,0\right) + 2d\left(|\vec{v}_2(z)|,t_1 - t_2\right) - d\left(|\vec{v}_1(z) + \vec{v}_2(z)|,t_1 - t_2\right)$$
$$- d\left(|\vec{v}_1(z) - \vec{v}_2(z)|,t_1 - t_2\right)\Big]\,dz \tag{B.15}$$

The integration over region (a) gives $\langle I\rangle^2$(1 + corrections) while the integration over region (b) gives $\langle I\rangle^2 e^{-D(1,2)}$(1 + corrections). As

indicated the corrections in region (b) are proportional to e^{-D} and are a small effect of no particular consequence. The corrections in region (a), on the other hand, are small but do not contain e^{-D} and hence fall much less rapidly. This leads to a coherence tail in $\langle I(1)I(2)\rangle$ which is not present in $\langle \mathcal{E}^*(1)\mathcal{E}(2)\rangle$. The interesting corrections in region (a) are computed by changing variables from $\vec{v}_1(z)$ and $\vec{v}_2(z)$ to $\vec{v}_1(z)$ and $\vec{w}(z) = \vec{v}_2(z) - \frac{z}{R}(\vec{r}_1 - \vec{r}_2) - \frac{R - z}{R}(\vec{r}_{o1} - \vec{r}_{o2})$ and then proceeding in exactly the same way as before. The result is

$$\langle I(1)I(2)\rangle = \langle I\rangle^2[1 + e^{-D(1,2)} + \gamma(\vec{r}_1 - \vec{r}_2, \vec{r}_{o1} - \vec{r}_{o2}, t_1 - t_2)] \tag{B.16}$$

where

$$\gamma(\vec{r}, \vec{r}_o, t) = 2\pi k^2 \int_0^R dz \int d^2q \exp\left[i \frac{z\vec{q} \cdot \vec{r}}{R} + i \frac{(R - z)\vec{q} \cdot \vec{r}_o}{R}\right] \tilde{\rho}(|\vec{q}|, t) Q(z, |\vec{q}|) \tag{B.17}$$

and Eq. (B.12) is not reproduced for $I(1) = I(2)$ because a small term of order γe^{-D} from region (b) has been dropped. For small α this integral can be simplified in the same way that Eq. (B.14) was obtained from Eq. (B.12). It becomes

$$\gamma(\vec{r}, \vec{r}_o, t) = \frac{\alpha\sqrt{3\pi}}{8} \frac{L \int_0^\infty q^2 \tilde{\rho}(q, t)[J_o(q|\vec{r}|) + J_o(q|\vec{r}_o|)]dq}{\int_0^\infty q\tilde{\rho}(q, 0)dq} \tag{B.18}$$

Corrections to more complicated correlations and terms of order α^2 or higher can also be computed — the only obstacle being the labor involved. The calculation of the general coherence tail involves only some combinatorics. It is

$$\frac{\left\langle \prod_{k=1}^{n} \left(I(k)\right)^{m_k} - \prod_{k=1}^{n} \left\langle I(k)\right\rangle^{m_k} \right\rangle}{\prod_{k=1}^{n} \left\langle I(k)^{m_k}\right\rangle} = \text{"Gaussian terms"} + \sum_{k,j=1}^{n} m_k m_j \gamma(k-j) \tag{B.19}$$

where $\gamma(k - j) = \gamma(\vec{r}_k - \vec{r}_j, \vec{r}_{ok} - \vec{r}_{oj}, t_k - t_j)$, the "Gaussian terms" are what one would compute from the Gaussian distribution and all terms of order e^{-D} have been dropped.

APPENDIX C: CORRECTIONS TO GAUSSIAN STATISTICS FOR $p < 2$

Eqs. (B.12), (B.13), (B.16) and (B.17) of App. (B) do not assume a single scale media and will be the starting point. For $p < 2$, in either the fully or the partially saturated regime Q can be approximated by

$$Q(z,|\vec{q}|) \approx \left(\frac{\vec{q}^{\,2}}{k} g(z,z)\right)^2 \exp\left[-\frac{\phi^2}{p+1}\left|\frac{|\vec{q}|\ g(z,z)}{kL}\right|^p\right] \tag{C.1}$$

where the cosine has been expanded, the short distance expansion for $\hat{\rho}$ Eq.(7.2) has been used and the identity

$$\int_0^R |g(z,z')|^p dz' = \frac{R}{p+1}|g(z,z)|^p \tag{C.2}$$

has been employed.

In the fully saturated regime the main contribution to Eqs. (B.12) and (B.17) again comes from $z \approx 0$ and $z \approx R$. Using Eq. (C.2) for Q then yields Eqs. (B.14) and (B.18) with α replaced by α' where α' is defined in Eq. (7.3).

In the partially saturated regime all values of z contribute to the integral but the dominant contribution comes from large $|q|$ where

$$\tilde{\rho}(q,t) = \frac{\hat{\rho}(0,t)2^p\ (\Gamma(1+\frac{1}{2}p))^2\ \sin(\pi p/2)}{4\pi^3\ |L|^p|q|^{p+2}} \tag{C.3}$$

For $n = 2$, Eq.(8.3) with

$$C(p) = \frac{(p+1)^{(4-p)/p}\ 2^p\ (\Gamma(1+\frac{1}{2}p)\Gamma(p-1))^2\ \sin(\pi p/2)\Gamma((4-p)/p)}{p\pi\ \ 6^{(2-p)}\ \ \Gamma(2p-2)} \tag{C.4}$$

is obtained by inserting Eqs.(C.3) (with t=0) and (C.1) in Eq.(B.12). The extension to general n works in the same way as before. The coherence tails in the partially saturated regime are obtained by inserting Eqs.(C.3) and (C.1) into Eq.(B.17). For $\vec{r}, \vec{r}_0 \neq 0$ this leads to integrals which cannot be done analytically.

APPENDIX D: THE PARTIALLY SATURATED REGIME FOR $p > 2$

It is difficult to make quantitative statements about the partially saturated regime when $4 > p > 2$. There is however some qualitative information.

Eq.(1.14) holds and $\langle\mathcal{E}\rangle$ is equal to $\mathcal{E}_0 \exp\left[-\frac{1}{2}\phi^2\right]$ in all regimes as long as the Markov approximation is valid. Furthermore $\left\langle\mathcal{E}^*(\omega')\,\mathcal{E}(\omega)\right\rangle$ continues to be given by Eqs.(1.19) - (1.21). The argument that any correlation involving an unequal number of $\mathcal{E}$'s and $\mathcal{E}^*$'s vanishes also goes through as before. Thus $\mathcal{E}$ is uniformly distributed in phase. The difficulty arises when one attempts to compute the non-vanishing higher moments.

The statistics are not Gaussian. This can be verified by assuming that they are and then computing the corrections. They are not small. Some information can be obtained however by comparing the path integral for $\left\langle(\mathcal{E}^*(2))^2\,(\mathcal{E}(1))^2\right\rangle$ with that for $\left\langle|\mathcal{E}(2)|^2\,|\mathcal{E}(1)|^2\right\rangle$. Upon doing this one finds that $\mathcal{E}$ always phase wraps as shown in Fig.(5b). It turns out that the typical space time scales over which the phase and intensity change are those listed in Table D-1 (the parameter δ was defined in Eq.(8.4)).

Phase	L/Φ	T/Φ	
ℓn I	$(L/\Phi)(\Omega/\Phi)^{\frac{p-2}{4-p}}$	$(T/\Phi)(\Omega/\Phi)^{\frac{2}{4-p}}$	$(T/\Phi)(\Omega/\Phi)^{\frac{p-\delta}{4-p}}$
		$p - \delta > 2$	$2 > p - \delta > 0$
	Scale Length	Scale Time	

Table D-1. Time and Space Scales Associated with Phase and ℓn I in the Partially Saturated Regime with $4 > p > 2$.

Note that for partial saturation where $\Phi^{4/p}/\Omega > 1$ but $\Phi/\Omega < 1$ the rate at which the intensity changes is always small compared to the rate at which the phase changes. Examining more complicated correlations leads to the conclusion that at a fixed frequency $\mathcal{E}$ can be represented as

$$\mathcal{E}(j) = \mathcal{E}_0(j) \exp\left[i\ \phi(j)\right] \chi(j) \tag{D.1}$$

where $\phi(j)$ is a real Gaussian random variable with $\langle \phi(j) \rangle = 0$ and

$$\left\langle (\phi(i) - \phi(j))^2 \right\rangle = D(i,j) \tag{D.2}$$

The other factor χ is an independent (of ϕ) complex random variable about which only three things are known:

(1) any correlation involving an unequal number of χ's and χ^*'s vanishes

(2) $<|\chi|^2> = 1$

and

(3) the decorrelation lengths and times for χ are those listed under intensity in Table D-1.

To see what the representation in Eq. (D.1) means consider

$$\frac{\langle \mathcal{E}^*(2)\,\mathcal{E}(1)\rangle}{\mathcal{E}_0^*(2)\,\mathcal{E}_0(1)} = \Big\langle \exp\left[i(\phi(1) - \phi(2))\right]\Big\rangle \Big\langle \chi^*(2)\,\chi(1)\Big\rangle$$

$$= \exp\left[-\tfrac{1}{2}D(1,2)\right] \langle |\chi|^2\rangle \qquad \text{(D.3)}$$

where to get the second line one notes that $\langle \chi^*(2)\ \chi(1)\rangle$ will be approximately $\langle |\chi|^2\rangle$ for all space or time separations such that $\exp\left[-\tfrac{1}{2}D\right]$ is not vanishingly small. Thus, Eq.(1.19) is reproduced, as it should be. Similarly,

$$\frac{\langle(\mathcal{E}^*(2))^2(\mathcal{E}(1))^2\rangle}{(\mathcal{E}_0^*(2))^2\ (\mathcal{E}_0(1))^2} = \exp\left[-2D(1,2)\right]\langle|\chi|^4\rangle \qquad \text{(D.4)}$$

and this correlation is known up to a constant. However, all that is known about the intensity correlation

$$\langle I(2)\ I(1)\rangle = \langle\ |\chi(2)|^2\ |\chi(1)|^2\rangle \qquad \text{(D.5)}$$

are its space and time scales.

The extension to unequal frequencies is straightforward. At different frequencies $\langle(\phi(\omega) - \phi(\omega'))^2\rangle = \left(\frac{\omega - \omega'}{\omega_g}\right)^2$ and $\langle \chi^*(\omega')\ \chi(\omega)\rangle = \Lambda(\omega - \omega')$. The higher order moments of χ are again non-Gaussian and unknown. However, their width in ω is large compared to ω_g. As in Sec.(8) this means that pulse propagation is dominated by wander rather than spreading.

As was mentioned in Sec.(8), there is a case where the non-Gaussian statistics of χ can be studied in detail. It is for correlations in time when $p = 2$ and $\delta = 0$ and is explained in Ref.(7).

The above results are most easily derived using the Fermat path formalism of Sec.6. One can work out the joint probability that two paths will satisfy the perturbed ray equation. In the partially saturated regime with $p > 2$ it turns out that the Fermat paths are highly correlated and tend to lie within $L(\Phi/\Omega)^{\frac{2}{4-p}}$ of each other. Studying averages of $\dot{\mu}$ and μ' along correlated Fermat paths then leads to the above conclusions. The detailed calculations are relatively straightforward but tedious and will not be given here.

APPENDIX E: CORRECTIONS TO THE MARKOV APPROXIMATION FOR INHOMOGENEOUS ANISOTROPIC MEDIA

If the $\vec{x}$ dependence of ρ is evaluated along the unperturbed ray then the first correction to the Markov approximation can be evaluated for a general homogeneous anisotropic medium. Let

$$\rho(\vec{x},\ t;\ \vec{s}(z) + \vec{e}_z z) = \int d^3\ell\, e^{i\vec{\ell}\cdot\vec{x}} \tilde{\rho}(\vec{\ell}, t; z) \tag{E.1}$$

then the generalization of Eq. (A.6) is

$$-\frac{\langle I\rangle - \langle I\rangle_0}{\langle I\rangle_0} = 2k^2 \int_0^R dz_1 \int_0^R dz_2 \int d^2q\ dq_z \tilde{\rho}\left(\vec{q} + \vec{e}_z q_z, 0;\ \tfrac{1}{2}(z_1 + z_2)\right)$$

$$\times \exp\left[i(q_z + \vec{q}\cdot\vec{s}\,'(\tfrac{1}{2}(z_1 + z_2))(z_1 - z_2)\right] \sin\left[\tfrac{1}{2}\vec{q}\cdot\vec{v}_0(z_1)\right]$$

$$\times \sin\left[\tfrac{1}{2}\vec{q}\cdot\vec{v}_0(z_2)\right] \exp\left[-\int_0^R d(\vec{v}_0(z), 0; z)dz\right] \tag{E.2}$$

where $q = (q_x, q_y)$ is a two dimensional vector, d is defined in Eq.(9.11) and

$$v_o(z)_i = \frac{q_j}{k}(g_{ij}(z, z_1) - g_{ij}(z, z_2)) \tag{E.3}$$

with g_{ij} defined in Eq.(9.22).

For an isotropic medium Eq.(E.2) can be analyzed in the same way as Eq.(A.6) and one finds that $(\langle I\rangle - \langle I\rangle_0)/\langle I\rangle_0$ is of order of the r.m.s. multiple scattering angle.

It is also straightforward to analyze Eq.(E.1) for a homogeneous but anisotropic medium. The result of doing this was stated in Sec.(9A).

APPENDIX F: CORRECTIONS TO GAUSSIAN STATISTICS FOR INHOMOGENEOUS ANISOTROPIC MEDIA

When the approximation of Eq.(9.7) for the correlation between two paths is made, it is possible to compute the corrections to Gaussian statistics in the saturated regimes. The calculation is a fairly straightforward generalization of that done in App.B and only the final result will be given.

Define a function $\tilde{\rho}_1(\vec{q};z)$ where $\vec{q} = (q_x, q_y)$ by $\tilde{\rho}_1(\vec{q},t;z) = \tilde{\rho}(\vec{q} - \vec{e}_z(\vec{s}\,'(z)\cdot\vec{q}),t;z)$ where $\tilde{\rho}$ is defined in Eq.(E.1). Then the analog of Eq.(B.12) is

$$\frac{\langle I^2\rangle - 2\langle I\rangle^2}{\langle I\rangle^2} = 4\pi k^2 \int_0^R dz \int d^2q\ \tilde{\rho}_1(\vec{q},0;z)\ Q_1(z,\vec{q}) \tag{F.1}$$

where

$$Q_1(z,\vec{q}) = 2\left[1 - \cos(q_i q_j g_{ij}(z,z)k^{-1})\right]\exp\left[-2\int_0^R d(\vec{v}_1(z,z'),0;z')dz'\right] \tag{F.2}$$

with

$$v_1(z,z')_i = k^{-1}q_j\ g_{ij}(z,z') \tag{F.3}$$

and d and g_{ij} are defined in Eqs.(9.11) and (9.22).

Using Ω and Φ as defined by Eqs.(9.21) and (9.6) it is possible to show that the right-hand side of Eq.(F.1) is small in the fully saturated regime and in the partially saturated regime for $p < 2$. As before, in the fully saturated regime the dominant contribution comes from the regions $z \approx 0$ and $z \approx R$ and in the partially saturated regime $\tilde{\rho}_1$ can be approximated

by its asymptotic form for large $|\vec{q}|$. Also, Q_1 can be simplified by expanding the cosine and replacing d by its expansion for small $\vec{v}_1$. The detailed calculation which is then fairly straightforward will be left to the reader.

The generalization to $\langle I^n \rangle$ works in the same way as in App.(B).

The coherence tail is given by Eq.(B.16) with

$$\gamma(\vec{r}_1-\vec{r}_2,\vec{r}_{01}-\vec{r}_{02},t)$$

$$= 2\pi k^2 \int_0^R dz \int d^2q \exp\left[i\,\vec{q}\cdot\vec{\boldsymbol{v}}(z)\right]\tilde{\rho}_1(\vec{q},t;z)Q_1(\vec{q},z) \tag{F.4}$$

where $\vec{\boldsymbol{v}}$ is defined in Eq.(9.14). The approximations mentioned above can also be made in the integral for γ. Finally, Eq.(B.19) holds with γ given by Eq.(F.4).

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5. Radio Science 10, (1975), Special Issue: Waves in Random Media.
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7. S. Flatté, ed., Sound Transmission through a Fluctuating Ocean, to be published by the Cambridge University Press.
8. The technical meaning of "a single scale" is that the three dimensional Fourier transform $\tilde{\rho}(\vec{q})$ of the covariance of μ should fall faster than $|\vec{q}|^{-6}$ at large $|\vec{q}|$. See Sec.(7).
9. The assumption that ρ's at different times are jointly Gaussian places restrictions on the dynamics of the medium. It is consistent with either the Taylor hypothesis (convection of a frozen field by a "wind") or time dependence due to linear wave motion.
10. Boundary conditions corresponding to, say, a plane wave at $z = 0$ can be obtained by superposition.
11. In deriving this equation one neglects $\frac{\partial^2}{\partial z^2}\mathcal{E}$ relative to $2ik\frac{\partial}{\partial z}\mathcal{E}$, μ^2 relative to 2μ and time derivatives of $\mathcal{E}$.

The latter requires the assumption that the medium does not change while a wave travels a distance L. This is true if condition (ii) (i.e., $kL << \omega T$) holds. Because R is always taken to be large compared to L, condition (iii) implies $<\mu^2>^{\frac{1}{2}} << 1$.

12. R. Mercier, Proc. Cambridge Phil. Soc. 58, 382 (1962).

13. The line $\Phi = \Omega$ is actually in a transition zone where the fluctuations are larger than Rayleigh, see Sec. (7).

14. Note that the r.m.s. multiple scattering angle is $6L/R\alpha$ so that α is restricted to be greater than $\sim 6L/R$. However, R is usually very large compared to L and there is no problem here.

15. In the optics literature (Refs. (4) and (5)) there is some controversy as to whether or not P(I) is Rayleigh in the saturated regime. Asymptotically it is but the corrections may be substantial in some experiments, see Secs. (7) and (8).

16. To make this rigorous, start with the finite form of the path integral in Eq. (2.1) and repeat the steps (including a summation by parts in the first term in the exponential) leading to Eq. (2.14) which is now an integral of finite dimension. Integrating over the discrete variables $\vec{u}_k$ will produce a product of δ-functions which force the $\vec{v}_k$ to satisfy $\vec{v}_{k-1} - 2\vec{v}_k + \vec{v}_{k+1} = 0$ with $\vec{v}_0 = \vec{r}_{01} - \vec{r}_{02}$ and $\vec{v}_n = \vec{r}_1 - \vec{r}_2$. There is a unique solution and in the continuum limit Eq.(2.16) is reproduced.

17. Up to a normalization it is the path integral for the unperturbed problem. The correct normalization is obtained by comparing both sides of the equation for the case where the fluctuations vanish and $d = 0$.

18. J. Lawson and G. Uhlenbeck, Threshold Signals, McGraw-Hill, New York (1950).

19. It is also implicitly assumed that the signal is in the saturated regime for all important frequencies in f.

20. This is M for region (a). In App.(B) it is shown that the integration over the center of gravity of all four paths produces a δ-functional which forces the difference between members of each pair to be equal ($\vec{v}_1$).

21. The case where there are many unperturbed rays is treated in detail in Ref.(7).

22. When multiplying vectors and matrices the summation convention (repeated indices are summed over) is used.

FIGURE CAPTIONS

FIG.1 The difference between multiple large angle scattering (a) and multiple small angle scattering (b).

FIG.2 Parameter regimes in $\Phi - \Omega$ space.

FIG.3 Illustrating Eq.(1.18). The signal η' will lie, with 90% probability, within the circles: (1) for $\Phi t/T$ small, (2) for $\Phi t/T \sim 1$ and (3) for $\Phi t/T$ large. The location of the signal η at $t=0$ was an unlikely one lying outside the 90% probability circle for a Rayleigh distribution.

FIG.4 A path in the path integral for $n=6$.

FIG.5 (a) The schematic track of a signal satisfying Gaussian statistics in time.

(b) The track of a signal which moves faster in phase than amplitude (phase wrapping).

FIG.6 Propagation through an anisotropic medium. The blobs are schematic inhomogeneities and the heavy directed lines are the unperturbed propagation path at various angles with respect to the long axis of the blobs.

FIG.7' Propagation in a channel. The channel axis (z-axis) is parallel to the long axis of the inhomogeneities (blobs). The medium is assumed to be isotropic in the z direction and the unperturbed propagation path (heavy directed line) makes periodic loops.

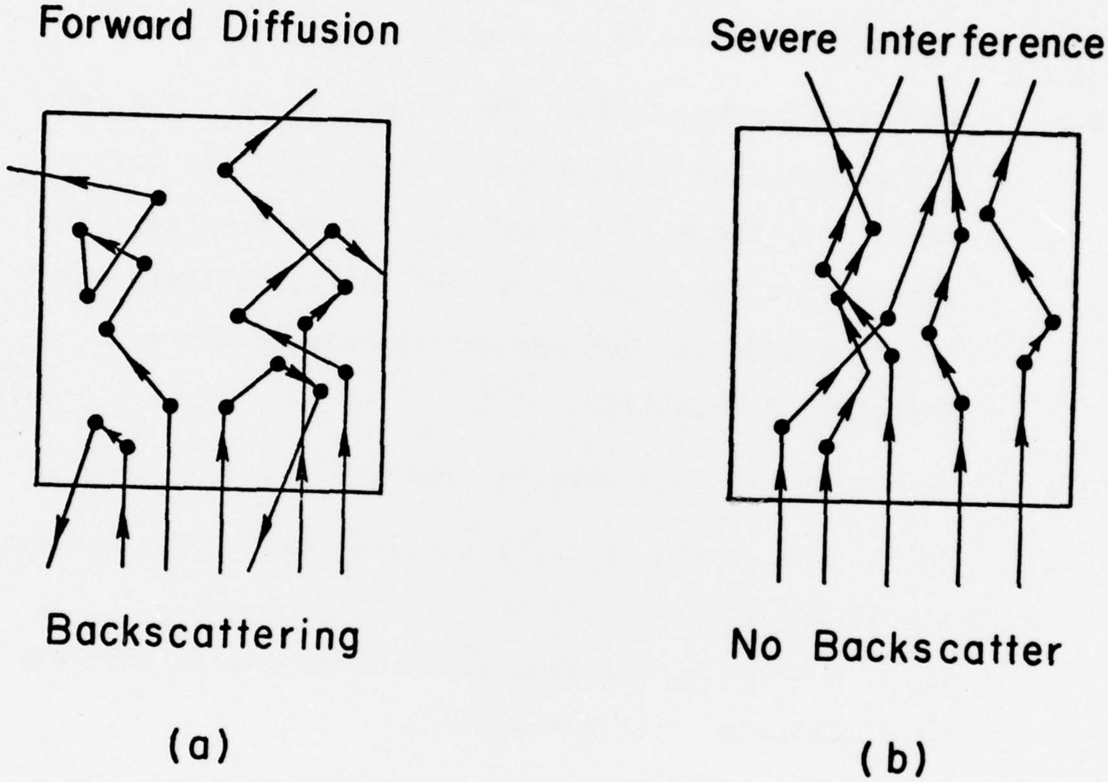

Forward Diffusion
Backscattering
(a)
Severe Interference
No Backscatter
(b)

FIG. 1

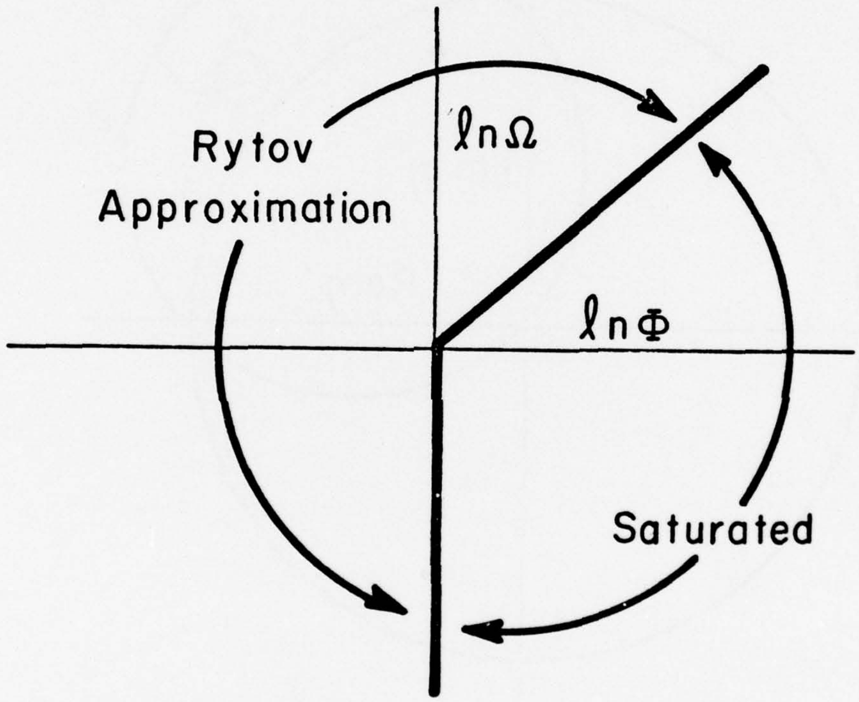


FIG. 2

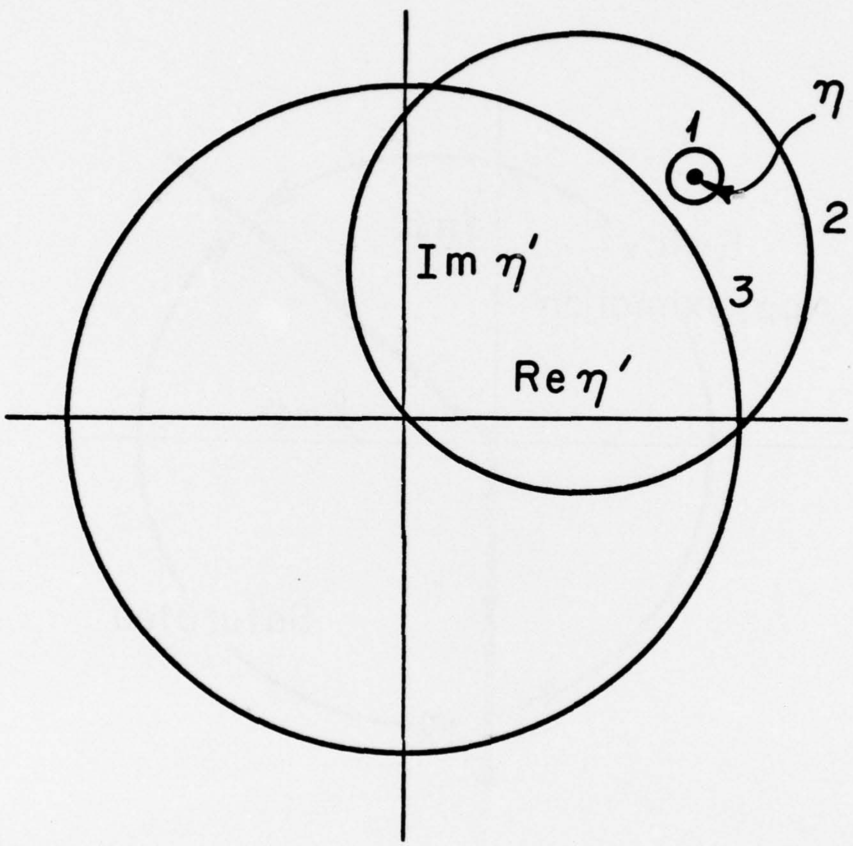


FIG. 3

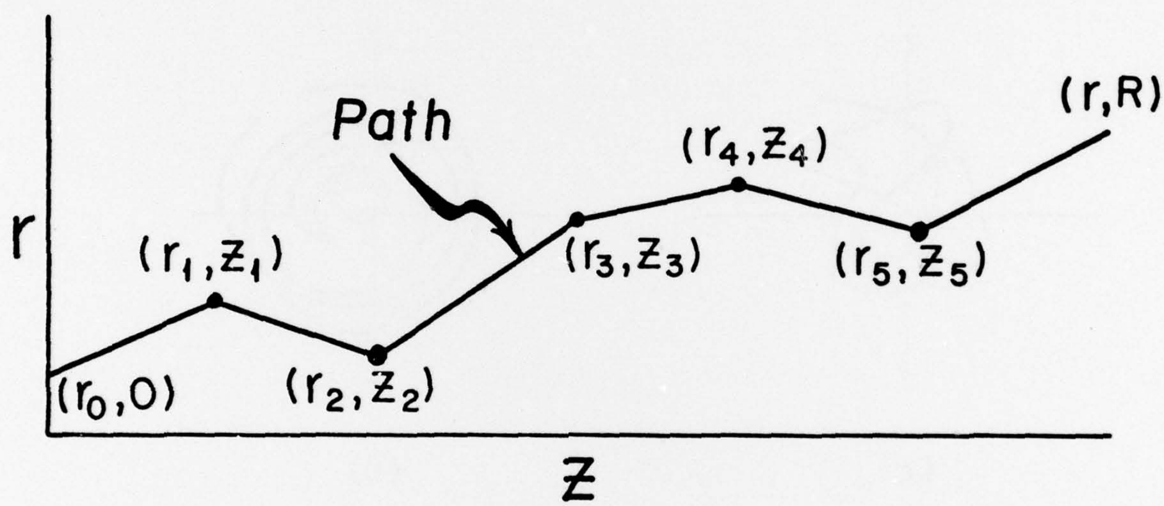
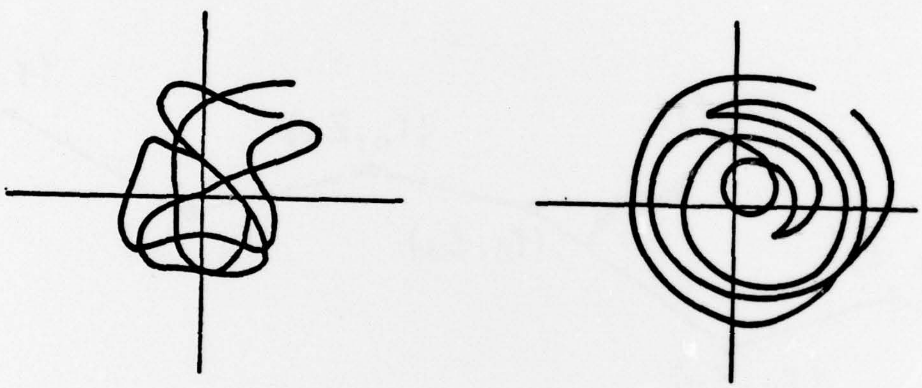


FIG. 4



(a) (b)

FIG. 5

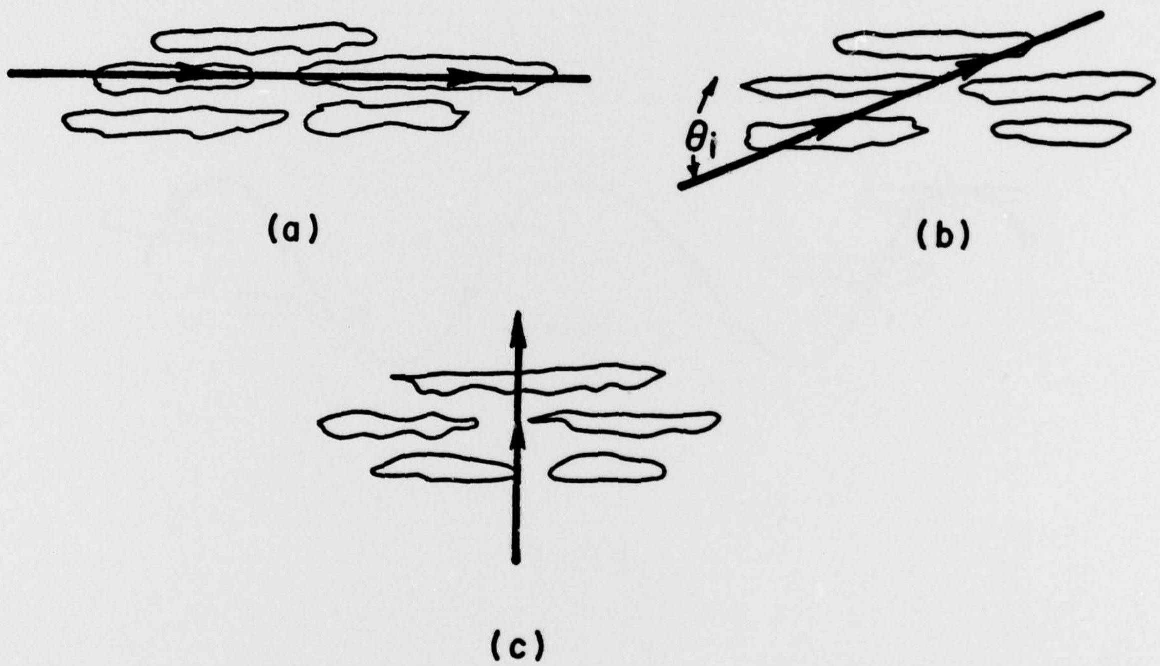


FIG. 6

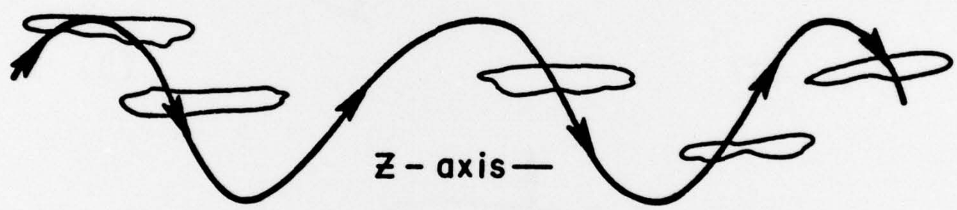


FIG. 7

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